Mathematics Department Stanford University
Math 51H Second Mid-Term, November 10, 2015

Solutions

Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ______________________

I understand and accept the provisions of the honor code (Signed) ______________________

1(a) (3 points.) (i) Give the definition of "U is open" and "C is closed" as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that if C_1, C_2 are closed then $C_1 \cup C_2$ is closed, and if U_1, U_2 are open then $U_1 \cap U_2$ is open.

Note: In (ii), at least one of the two statements should be shown directly from the definition. You may either show the other directly, or by using an appropriate theorem.

Solution: (i) U open means that for each $y \in U$ there is a $\rho > 0$ such that $B_\rho(y) \subset U$. C closed means that C contains all its limit points. That is if $\{x_k\}$ is a convergent sequence in $\mathbb{R}^n$ and $x_k \in C$ for each k, then $\lim x_k \in C$.

(ii) If U_1, U_2 are open and $a \in U_1 \cap U_2$ then $a \in U_j$, $j = 1, 2$, so by the openness of U_j there is $\rho_j > 0$ such that $B_{\rho_j}(a) \subset U_j$. Let $\rho = \min(\rho_1, \rho_2) > 0$, so $B_\rho(a) \subset B_{\rho_j}(a) \subset U_j$ for $j = 1, 2$, and thus $B_\rho(a) \subset U_1 \cap U_2$, proving the openness of $U_1 \cap U_2$.

This implies that if C_1, C_2 are closed then $C_1 \cup C_2$ is closed, since by the theorem in lecture, a set is closed iff its complement is open. Thus, $(C_1 \cup C_2)^c = C_1^c \cap C_2^c$ shows that $(C_1 \cup C_2)^c$ is open by what we have shown, and thus $C_1 \cup C_2$ closed by the just stated theorem from lecture.

Alternatively, suppose $\{x_k\}$ is a sequence in $C_1 \cup C_2$ converging to some $x \in \mathbb{R}^n$. Then for each k, $x_k \in C_1$ or $x_k \in C_2$, so with K_j, $j = 1, 2$, the set of k such that $x_k \in C_j$, $K_1 \cup K_2 = \mathbb{N}^+$, and thus one of K_j is infinite. Let i be such that K_i is infinite, and consider the subsequence $\{x_{k_m}\}_{m=1}^\infty$ of $\{x_k\}$ containing exactly the elements of $\{x_k\}$ with $k \in K_i$. Then $\{x_{k_m}\}_{m=1}^\infty$ is a sequence in C_i, converges to x (being a subsequence of sequence so converging), so by the closedness of C_i, $x \in C_i$, and thus $x \in C_1 \cup C_2$, showing the claimed closedness.

1(b) (3 points) (i) For $U \subset \mathbb{R}^n$ open, give the definition of $f : U \to \mathbb{R}^k$ being continuous, and (ii) show that if $f : U \to V \subset \mathbb{R}^k$ is continuous, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ are open, $g : V \to \mathbb{R}^m$ is continuous then $g \circ f$, defined by $(g \circ f)(x) = g(f(x))$, is continuous.

Solution: (i) f is continuous if for all $a \in U$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - a\| < \delta$, $x \in U$ implies $\|f(x) - f(a)\| < \varepsilon$.

(ii) Suppose f, g are as stated, and let $a \in U$, so $f(a) \in V$. Let $\varepsilon > 0$. By the continuity of g there exists $\delta' > 0$ such that $\|y - f(a)\| < \delta'$, $y \in V$ implies $\|g(y) - g(f(a))\| < \varepsilon$. But then by the definition of continuity of f, applied with δ', there exists $\delta > 0$ such that $\|x - a\| < \delta$, $x \in U$ implies $\|f(x) - f(a)\| < \delta'$. Thus, $\|x - a\| < \delta$, $x \in U$ implies $\|f(x) - f(a)\| < \delta'$ which in turn implies $\|g(f(x)) - g(f(a))\| < \varepsilon$, showing the claimed continuity.

2(a) (3 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{7}(x^7 + y^7) - 64x - y$. Find all the critical points (i.e. points where $\nabla_{\mathbb{R}^2} f = 0$) of f, and discuss whether these points are local max/min for f. Justify all claims either with proof or by using a theorem from lecture.

Solution: $Df(x, y) = (x^6 - 64, y^6 - 1)$, so there are 4 critical points $(2, 1), (-2, -1), (2, -1), (-2, 1)$. The Hessian matrix at (x, y) is $\begin{pmatrix} 6x^5 & 0 \\ 0 & 6y^5 \end{pmatrix}$ which gives positive definite quadratic form $6 \cdot 32\lambda^2 + 6\mu^2$ at $(2, 1)$ and negative definite quadratic form $-6 \cdot 32\lambda^2 - 6\mu^2$ at $(-2, -1)$. Hence by the Second Derivative test from lecture (applicable because f is C^2, in fact C^∞), we see that f has a a local minimum at $(2, 1)$ and a local maximum at $(-2, -1)$. At the point $(-2, 1)$ the Hessian quadratic form is $-6 \cdot 32\lambda^2 + 6\mu^2$ which changes sign (has positive max on S^1 and a negative min on S^1), and hence, as we proved in lecture/section, it is neither a local max nor a local min for f. (Concretely, $f(x, 1)$ has a local max at -2, $f(-2, y)$ has a local min at $y = 1$.) Similarly the point $(2, -1)$ is neither a local max nor a local min for f.

2(b) (3 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 1 + 3x^2 + y^6 + 4(x - 1)^4$. Show that f is bounded below and it attains its minimum.

Note: you do not need to find where the minimum is attained. Hint: show first that if $|x| \geq 3$ or $|y| \geq 2$ then $f(x, y) \geq 65$. What is $f(0, 0)$?

Solution: Since all terms in the expression for f are squares of real numbers, we have $f(x, y) \geq 1$, so f is bounded below. Moreover, if $|x| \geq 3$ then $|x-1| \geq |x|-1 \geq 2$ (since $|x| \leq |x-1|+1$ by the triangle inequality) so $f(x, y) \geq 1 + 4 \cdot 16 = 65$ (using that all other terms are ≥ 0). If $|y| \geq 2$ then $f(x, y) \geq 1 + 64 = 65$ (again using that all other terms are ≥ 0). Thus, if $|x| \geq 3$ or $|y| \geq 2$ then $f(x, y) \geq 65$. On the other hand $R = \{(x, y) : |x| \leq 3, |y| \leq 2\}$ is a closed and bounded subset of $\mathbb{R}^2$; it is bounded directly from the definition and closed because it is the intersection of the inverse images of the closed intervals $[-3, 3]$ resp. $[-2, 2]$ under the continuous maps $g(x, y) = x$ and $h(x, y) = y$, i.e. it is the intersection of two closed sets, thus closed. Correspondingly, by the theorem in lecture, $R = \{(x, y) : |x| \leq 3, |y| \leq 2\}$ is compact, and as f is continuous, $f|_R$ attains its minimum there, say at the point (x_0, y_0). Note that as $f(0, 0) = 1 + 4 = 5$ and $(0, 0) \in R$, the minimum value $f(x_0, y_0) \leq 5 < 65$. Since $f(x, y) \geq 65$ when $(x, y) \notin R$, we conclude that the minimum of f over $\mathbb{R}^2$ (and not just R!) is indeed attained at (x_0, y_0).

3(a) (3 points) Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. (i) Find its radius of convergence ρ. (ii) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $|x| < \rho$. Show that $f'(x) = \frac{1}{1-x}$ for $|x| < \rho$.

Solution: (i) First, recall that the series $\sum_{n=1}^{\infty} 1/n$ diverges, and this is just the power series evaluated at 1, so as a power series converges absolutely in $(-\rho, \rho)$, if ρ is its radius of convergence, we must have $\rho \leq 1$. On the other hand, $|x^n/n| \leq |x^n|$, and $\sum_{n=1}^{\infty} |x^n|$ converges for x with $|x| < 1$ (this being a geometric series with common ratio $|x|$), by the comparison theorem for series with non-negative terms (i.e. the convergence theorem for increasing sequences which are bounded above), $\sum_{n=1}^{\infty} |x^n/n|$ converges for $|x| < 1$, thus (absolute convergence implies convergence) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $|x| < 1$. Hence the radius of convergence is ≥ 1, so in summary $\rho = 1$.

(ii) By the theorem from class, a power series is infinitely differentiable within its radius of convergence with derivatives given by term-by-term differentiation. Hence, for $|x| < 1$, $f'(x)$ exists and is $f'(x) = \sum_{n=1}^{\infty} n \frac{x^{n-1}}{n} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, where the last equality comes from the sum of a convergent geometric series.

3(b) (3 points): (i) A sequence of functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}^+$ such that $n \geq N$ implies that $\sup\{|f_n(x) - f(x)| : x \in [a, b]\} < \varepsilon$. Show that if f_n are continuous and $f_n \to f$ uniformly then f is continuous.

Hint: continuity of f at x requires given $x \in [a, b]$ and $\varepsilon > 0$ finding $\delta > 0$ with certain properties. Express $|f(y) - f(x)|$ in terms of $|f_n(y) - f_n(x)|$ and other expressions, and choose n well.

Solution: Suppose f_n continuous for all n, f_n converges to f uniformly. We need to show that f is continuous. So let $x \in [a, b]$ and $\varepsilon > 0$. For any $y \in [a, b]$ and any n we have

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

by the triangle inequality. So first choose n such that the first and the last terms are guaranteed to be small, namely choose n such that $\sup\{|f_n(x) - f(x)| : x \in [a, b]\} < \varepsilon/3$, we can do this due to the uniform convergence of f_n to f. Then the first and last terms are $< \varepsilon/3$. Now for this n, using the continuity of f_n at x, we get $\delta > 0$ such that $|y - x| < \delta$, $y \in [a, b]$ implies $|f_n(y) - f_n(x)| < \varepsilon/3$. Thus, $|y - x| < \delta$, $y \in [a, b]$ implies $|f(y) - f(x)| < \varepsilon$, which proves that f is continuous, completing the proof.

4(a) (3 points.) (i) Give the definition of a curve $\gamma : [a,b] \to \mathbb{R}^n$ having finite length, and for curves of finite length state the definition of the "length of a curve $\gamma : [a,b] \to \mathbb{R}^n$." (ii) Show that if $\gamma : [a,b] \to \mathbb{R}^n$ has the property that there is a constant $K > 0$ such that $\|\gamma(t) - \gamma(t')\| \leq K|t - t'|$ for $t, t' \in [a,b]$ (one says γ is Lipschitz) then γ has finite length.

Solution: (i) A curve (a continuous map) $\gamma : [a,b] \to \mathbb{R}^n$ has finite length if the set $\{\ell(\gamma, \mathcal{P}) : \mathcal{P} \text{ partition of } [a,b]\}$ is bounded above, in which case $\ell(\gamma)$ is the supremum of this set. Here $\ell(\gamma, \mathcal{P}) = \sum_{j=1}^N \|\gamma(t_j) - \gamma(t_{j-1})\|$, where $\mathcal{P}$ is the partition $a = t_0 < t_1 < \ldots < t_N = b$.

(ii) Suppose γ is as above. For any partition $\mathcal{P}$ of $[a,b]$, say $a = t_0 < t_1 < \ldots < t_N = b$, we have

$$\ell(\gamma, \mathcal{P}) = \sum_{j=1}^N \|\gamma(t_j) - \gamma(t_{j-1})\| \leq \sum_{j=1}^N K|t_j - t_{j-1}| = \sum_{j=1}^N K(t_j - t_{j-1}) = K(t_N - t_0) = K(b-a).$$

Thus $\{\ell(\gamma, \mathcal{P}) : \mathcal{P} \text{ partition of } [a,b]\}$ is bounded above, with $K(b-a)$ being an upper bound, and correspondingly γ has finite length; in fact $\ell(\gamma) \leq K(b-a)$.

4(b) (4 points.) (i) Show directly (without using the corollary of the implicit function theorem that we have not proved) that the set $M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 + 1\}$ is a 2-dimensional C^1 submanifold of $\mathbb{R}^3$. (ii) Find the tangent space of M at the point $(1,1,1)$, and give a basis for it.

Note: in fact, M is a C^∞ submanifold. You may use that $\sqrt{\ } : (0,\infty) \to (0,\infty)$ is C^∞.

Solution: (i) It is often convenient to use the notation (x_1, x_2, x_3) below. By the equivalent statement to the definition discussed in section, for each point $a \in M$, we need to find an open set $V \subset \mathbb{R}^3$ containing it, a permutation map P, an open subset U of $\mathbb{R}^2$ and a C^1 map g such that $V \cap M = P(G(U))$, where $G(x_1, x_2) = (x_1, x_2, g(x_1, x_2))$. This is equivalent to saying that one of the coordinates x, y, z has to be expressed as a graph over an open subset U of the remaining coordinates' plane. We can write $M = M_{1,+} \cup M_{1,-} \cup M_{2,+} \cup M_{2,-} = \cup_{j=1,2} \cup_{\pm} M_{j,\pm}$, where $M_{j,\pm} = \{(x_1, x_2, x_3) \in M : \pm x_j > 0\}$. Indeed, certainly $M_{j,\pm} \subset M$ for all j and $\pm$, and conversely if $(x_1, x_2, x_3) \in M$ then $x_1^2 + x_2^2 \geq 1$, so at least one of x_1 and x_2 is nonzero, thus either positive or negative, so the point is in one of $M_{j,\pm}$. Let $V_{j,\pm} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \pm x_j > 0\}$; this is open being the inverse image of the open set $(0,\infty)$ under the map $h_{j,\pm}(x_1, x_2, x_3) = \pm x_j$; then $M \cap V_{j,\pm} = M_{j,\pm}$. Thus, it suffices to show that $M_{j,\pm}$ is the image of a permuted graph map. For the sake of definiteness, consider $M_{1,+}$; all others are similar. Points in $M_{1,+}$ satisfy $x_1 > 0$ and $x_1^2 + x_2^2 = x_3^2 + 1$, thus $x_2^2 < x_3^2 + 1$, i.e. $|x_2| < \sqrt{x_3^2 + 1}$, and $x_1 = \sqrt{x_3^2 + 1 - x_2^2}$, with all square roots being the non-negative square roots of non-negative reals. Now the set $U_{1,+} = \{(x_2, x_3) : x_2^2 < x_3^2 + 1\} \subset \mathbb{R}^2$ is open, being the inverse image of $(0,\infty)$ under the continuous map $h(x_2, x_3) = x_3^2 + 1 - x_2^2$, and $M_{1,+}$ is the permuted graph of the C^∞ function $g_{1,+}(x_2, x_3) = \sqrt{x_3^2 + 1 - x_2^2}$ over $U_{1,+}$, with the C^∞ statement due to being the composition of C^∞ functions, $\sqrt{\ }$ defined over $(0,\infty)$, and a polynomial. This, together with completely analogous considerations for the other $M_{j,\pm}$ proves that M is a 2-dimensional C^∞ submanifold of $\mathbb{R}^3$.

(ii) Notice that $(1,1,1) \in M_{1,+}$, so by the theorem in lecture the tangent space to M at $(1,1,1)$ is the span of the partial derivatives of the graph map, with the latter being linearly independent and thus forming a basis. Concretely, the permuted graph map is $\tilde{G}(x_2, x_3) = (\sqrt{x_3^2 + 1 - x_2^2}, x_2, x_3)$, $(x_2, x_3) \in U_{1,+}$, so a basis of the tangent space at $\tilde{G}(x_2, x_3)$ is given by

$$(-x_2/\sqrt{x_3^2 + 1 - x_2^2}, 1, 0)^{\mathrm{T}}, (x_3/\sqrt{x_3^2 + 1 - x_2^2}, 0, 1)^{\mathrm{T}},$$

i.e. at $(1,1,1)$ (corresponding to $\tilde{G}(1,1)$) by $(-1,1,0)^{\mathrm{T}}, (1,0,1)^{\mathrm{T}}$. Note that these vectors are indeed orthogonal to the gradient of $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 - 1$, which is $\nabla f = (2x_1, 2x_2, -2x_3)^{\mathrm{T}}$, i.e. is $(1,1,-1)^{\mathrm{T}}$ at $(1,1,1)$, thus their span (being 2-dimensional) is exactly the orthocomplement of the span of ∇f.