Mathematics Department Stanford University
Math 51H Second Mid-Term, November 12, 2013

75 MINUTES

Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ________________________

I understand and accept the provisions of the honor code (Signed) ________________________

1(a) (3 points) State the chain rule for the composite function $g \circ f$, where $f : U \to V$ and $g : V \to \mathbb{R}^p$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open. Using the chain rule, or otherwise, prove that if $g : \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n$, if $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^n$, and if $h(t) = g(\underset{\sim}{a} + t\underset{\sim}{b})$ for $t \in \mathbb{R}$, then $h'(0)$ exists, and find its value in terms of the components $b_1, \ldots, b_n$ of $\underset{\sim}{b}$ and the partial derivatives of g at $\underset{\sim}{a}$.

1(b) (3 points.) (i) Give the definition of "U is open" and "C is closed" as applied to subsets $U, C \subset \mathbb{R}^n$, and (ii) give the proof that $\mathbb{R}^n \setminus U$ closed implies U open.

Note: In lecture we proved $\mathbb{R}^n \setminus U$ is closed $\iff$ U is open; in (ii) you are only being asked to prove "$\Rightarrow$."

2(a) (3 points): Suppose $\delta > 0$ and $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ are convergent power series in $(-\delta, \delta)$. Prove $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for each $x \in (-\delta, \delta)$ implies that $a_n = b_n$ for each $n = 0, 1, 2, \ldots$.

Hint: Since we can take $c_n = a_n - b_n$, it suffices to prove $\sum_{n=0}^{\infty} c_n x^n = 0 \, \forall \, x \in (-\delta, \delta) \Rightarrow c_n = 0 \, \forall \, n = 0, 1, 2, \ldots$.

2(b) (3 points.) (i) Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is AC on all of $\mathbb{R}$.

(ii) If we define $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, prove that $\exp(x + t) = (\exp x)(\exp t)$.

Hint for (ii): For fixed t let $f(x) = \exp(x + t)$ and $g(x) = (\exp t)(\exp x)$. Start by checking that $f^{(n)}(0) = g^{(n)}(0)$ for each $n = 0, 1, 2, \ldots$.

3(a) (4 points.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{3}y^3 + xy + x^2$. Find the critical points (i.e. points where $\nabla_{\mathbb{R}^2} f = \underline{0}$) of f, and state whether each critical point is a local max, local min or neither. Make sure you justify all claims you make in your argument, either with a proof or by quoting the appropriate theorem from lecture.

3(b) (3 points.) Give the definition of "length of a curve $\gamma : [a, b] \to \mathbb{R}^n$." Using any theorem from lecture that you need, find the length of γ in case $n = 2$ and $\gamma(t) = (\sin t^2, \cos t^2), t \in [0, 2]$.

4(a) (3 points.) Prove that $M = \{\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3^2 = 1 + x_1^2 + x_2^2\}$ is a C^1 manifold, and find the tangent space $T_{\underline{a}}M$ at the point $\underline{a} = (2, 2, -3)$.

Note: You should give a basis for the tangent space.

4(b) (3 points.) (i) If M is a k-dimensional C^1 submanifold of $\mathbb{R}^n$ ($n \geq 2$ and $1 \leq k \leq n-1$ given), and $f : W \to \mathbb{R}$ is C^1 with $W \subset \mathbb{R}^n$ open, $W \supset M$, give the definition of "the tangential gradient $\nabla_M f$" and "a critical point of $f|M$." (ii) In the special case when $M = S^{n-1}$ (so $k = n-1$ and f is C^1 on an open set $W \supset S^{n-1}$) prove that $f|S^{n-1}$ has at least two distinct critical points $\underline{a}, \underline{b} \in S^{n-1}$.