Mathematics Department Stanford University
Math 51H Mid-Term 1

October 13, 2015

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ______________________

I understand and accept the provisions of the honor code (Signed) ______________________

1 (a) (3 points): (i) Give the ε, N definition of "$\lim a_n = \ell$," where $\{a_n\}_{n=1,2,\ldots}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and (ii) use your definition to prove that if $\{a_n\}_{n=1,2,\ldots}$ converges to $\ell \neq 0$, then there exists N such that $|a_n| > |\ell|/2$ for $n \geq N$.

Note for (ii): You may not use any of our limit theorems to prove (ii), only the definition of the limit, and properties of the reals.

1(b) (3 points): Suppose that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. Let $s_k = \sup\{a_n : n \geq k\}$, $k \in \mathbb{N}^+$, i.e. s_k is the sup of all but the first $k-1$ elements of the original sequence. Show that $\lim s_k$ exists.

Note: You should in particular explain why s_k itself exists. One writes $\limsup a_n = \lim s_k$; this gives a measure how large a_n can be for large n.

2(a) (3 points): (i) Give the definition of the orthocomplement $V^{\perp}$ of a subspace V of an inner product space Z (if you wish, you way assume $Z = \mathbb{R}^n$ with usual inner product) and (ii) show that if $\{v_1, \ldots, v_k\}$ is a basis of a subspace V of Z (again $Z = \mathbb{R}^n$ may be assumed), then $V^{\perp} = \{w \in Z : \ w \cdot v_j = 0, \ j = 1, 2, \ldots, k\}$.

2(b) (4 points): Suppose V is a vector space (if you wish you may assume that it is a subspace of $\mathbb{R}^n$), $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k \in V$ and $V = \text{span}\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k\}$. Show that there is a sub-collection $\{\underset{\sim}{v}_{i_1}, \underset{\sim}{v}_{i_2}, \ldots, \underset{\sim}{v}_{i_l}\}$, $i_1 < i_2 < \ldots < i_l$ (possibly $l = 0$), such that $\{\underset{\sim}{v}_{i_1}, \underset{\sim}{v}_{i_2}, \ldots, \underset{\sim}{v}_{i_l}\}$ is a basis for V.

Hint for (b): Analogously to the proof of the basis theorem, consider a minimal size subcollection that spans V, or a maximal size subcollection which is linearly independent.

3(a) (3 points): (i) State the rank nullity theorem. (ii)-(iii): Suppose A is an $m \times n$ matrix and $C(A) = \mathbb{R}^m$. (ii) Show that $m \leq n$. (iii) If in addition $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for every $\underset{\sim}{b} \in \mathbb{R}^m$, show that $m = n$.

3(b) (3 points): (i) Find the matrices A_1, A_2 of the orthogonal projections P_{V_j}, $j = 1, 2$, to $V_1 = \text{Span}\{(1,1,1)^{\mathrm{T}}\}$ and $V_2 = \text{Span}\{(1,-1,0)^{\mathrm{T}}\}$ in $\mathbb{R}^3$. (ii) Show that the matrix of the orthogonal projection P_V to $V = \text{Span}\{(1,1,1)^{\mathrm{T}}, (1,-1,0)^{\mathrm{T}}\}$ is $A_1 + A_2$.

Hint for (ii): Note that $(1,1,1)^{\mathrm{T}}$ and $(1,-1,0)^{\mathrm{T}}$ are orthogonal.

4 (6 points): Find (i) rref A (showing all row operations), (ii) a basis for the null space $N(A)$, (iii) a basis for the column space of A and (iv) $\dim N(A^{\mathrm{T}})$, if

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ -1 & -2 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

.

.