Mathematics Department Stanford University
Math 51H Mid-Term 1

October 13, 2015

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ____________________

I understand and accept the provisions of the honor code (Signed) ____________________

1 (a) (3 points): (i) Give the ε, N definition of "$\lim a_n = \ell$," where $\{a_n\}_{n=1,2,\ldots}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and (ii) use your definition to prove that if $\{a_n\}_{n=1,2,\ldots}$ converges to $\ell \neq 0$, then there exists N such that $|a_n| > |\ell|/2$ for $n \geq N$.

Note for (ii): You may not use any of our limit theorems to prove (ii), only the definition of the limit, and properties of the reals.

Solution: $\lim a_n = \ell$ means that for each $\varepsilon > 0$ there is N such that $|a_n - \ell| < \varepsilon$ for all $n \geq N$.

Now suppose that $\lim a_n = \ell$, $\ell \neq 0$. Then there exists N such that for $n \geq N$, $|a_n - \ell| < |\ell|/2$; here we use that $\varepsilon = |\ell|/2 > 0$ since $\ell \neq 0$. Then for $n \geq N$, $|a_n| = |(a_n - \ell) + \ell| \geq |\ell| - |a_n - \ell|$ by the triangle ineqality (use $x = a_n$, $y = \ell - a_n$, so $|x + y| \leq |x| + |y|$ is $|\ell| \leq |a_n| + |\ell - a_n|$, and rearrange). Thus, for $n \geq N$, $|a_n| > |\ell| - |\ell|/2 = |\ell|/2$, as desired.

1(b) (3 points): Suppose that $\{a_n\}_{n=1}^\infty$ is a bounded sequence. Let $s_k = \sup\{a_n : \ n \geq k\}$, $k \in \mathbb{N}^+$, i.e. s_k is the sup of all but the first $k-1$ elements of the original sequence. Show that $\lim s_k$ exists.

Note: You should in particular explain why s_k itself exists. One writes $\limsup a_n = \lim s_k$; this gives a measure how large a_n can be for large n.

First note that as $\{a_n : \ n \geq k\} \subset \{a_n : \ n \in \mathbb{N}^+\}$, and the latter is bounded by assumption, so is the former. Moreover, the former is non-empty as a_k is in it, thus its supremum exists by the completeness property of the reals.

We claim that $\{s_k\}_{k=1}^\infty$ is a decreasing sequence and it is bounded below, and thus by the theorem from the lecture/book/HW, it converges. To see that $s_k \geq s_{k+1}$, note that with $S_k = \{a_n : \ n \geq k\}$, $S_{k+1} \subset S_k$. Thus, any upper bound for S_k is an upper bound for S_{k+1}, in particular $s_k = \sup S_k$ is such. Since $s_{k+1} = \sup S_{k+1}$ is the least upper bound for S_{k+1}, we have $s_{k+1} \leq s_k$, as desired. Now, to see that $\{s_k\}_{k=1}^\infty$ is bounded below, let C be a lower bound for $S_1 = \{a_n : \ n \in \mathbb{N}^+\}$ so $a_n \geq C$ for all n. Thus, for every k, $a_k \in S_k$ shows that $s_k \geq a_k \geq C$ so C is a lower bound for the sequence $\{s_k\}_{k=1}^\infty$. As already explained, this completes the proof of the convergence of $\{s_k\}_{k=1}^\infty$.

2(a) (3 points): (i) Give the definition of the orthocomplement $V^\perp$ of a subspace V of an inner product space Z (if you wish, you way assume $Z = \mathbb{R}^n$ with usual inner product) and (ii) show that if $\{v_1, \ldots, v_k\}$ is a basis of a subspace V of Z (again $Z = \mathbb{R}^n$ may be assumed), then $V^\perp = \{w \in Z : \ w \cdot v_j = 0, \ j = 1, 2, \ldots, k\}$.

Solution: The orthocomplement $V^\perp$ is the set

$$V^\perp = \{w \in Z : \ \forall v \in V, w \cdot v = 0\};$$

note that $V^\perp$ is a subspace of Z.

Let $\{v_1, \ldots, v_k\}$ be a basis for V. We now show that

$$V^\perp = \{w \in Z : \ w \cdot v_j = 0, \ j = 1, 2, \ldots, k\}.$$

Indeed, certainly if $w \in V^\perp$ then $w \cdot v_j = 0$ for all j, giving the containment $\subset$. Conversely, if $w \cdot v_j = 0$ for all j, then $w \cdot \sum_{j=1}^k c_j v_j = 0$ for all $c_j \in \mathbb{R}$ by the linearity of the inner product in its second slot. But any $v \in V$ can be written as $v = \sum_{j=1}^k c_j v_j$ since the v_j form a basis of V, so $w \cdot v = 0$ for all $v \in V$, thus $w \in V^\perp$. This shows the containment $\supset$, and thus the claimed equality.

2(b) (4 points): Suppose V is a vector space (if you wish you may assume that it is a subspace of $\mathbb{R}^n$), $\underline{v}_1, \ldots, \underline{v}_k \in V$ and $V = \text{span}\{\underline{v}_1, \ldots, \underline{v}_k\}$. Show that there is a sub-collection $\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\}$, $i_1 < i_2 < \ldots < i_l$ (possibly $l = 0$), such that $\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\}$ is a basis for V.

Hint for (b): Analogously to the proof of the basis theorem, consider a minimal size subcollection that spans V, or a maximal size subcollection which is linearly independent.

Solution: Let

$$S = \{l \in \{0, 1, \ldots, k\} : \ \exists i_1 < i_2 < \ldots < i_l \text{ s.t. } \text{span}\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\} = V\}.$$

Then $k \in S$ since $i_j = j$ for all $j = 1, \ldots, k$ gives a spanning set, so S is a non-empty set of positive integers. Correspondingly it has a smallest element, call it l. Then there exists $i_1 < i_2 < \ldots < i_l$ such that $\text{span}\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\} = V$. We claim that $\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\}$ is a basis of V; since it spans, we just need to show it is linearly independent. Note that if $l = 0$, the collection is the empty collection, and is thus linearly independent by definition, and $V = \{0\}$. So suppose $l \geq 1$, and $\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\}$ is not linearly independent. Then as shown in lecture/book there exists m such that v_{i_m} is a linear combination of the remaining v_{i_j}: $v_{i_m} = \sum_{j \neq m} c_j v_{i_j}$ for some scalars c_j. In particular, any vector v in the span of $\{\underline{v}_{i_1}, \underline{v}_{i_2}, \ldots, \underline{v}_{i_l}\}$, so $v = \sum_{r=1}^l d_{i_r} v_{i_r}$ for some scalars d_{i_r}, is also in the span of the remaining v_{i_j} (with v_{i_m} dropped), by substituting in the linear combination $v_{i_m} = \sum_{j \neq m} c_j v_{i_j}$ into $v = \sum_{r=1}^l d_{i_r} v_{i_r}$. Thus, $l - 1 \in S$ since the remaining $l - 1$ vectors v_{i_j} satisfy all requirements for $l - 1$ to be in S. But this contradicts that l is the smallest element of S.

3(a) (3 points): (i) State the rank nullity theorem. (ii)-(iii): Suppose A is an $m \times n$ matrix and $C(A) = \mathbb{R}^m$. (ii) Show that $m \leq n$. (iii) If in addition $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for every $\underset{\sim}{b} \in \mathbb{R}^m$, show that $m = n$.

Solution: (i) The rank nullity theorem is that for an $m \times n$ matrix A, $\dim N(A) + \dim C(A) = n$. Alternatively, for a linear map $T : V \to W$ with V finite dimensional, $\dim V = \dim N(T) + \dim \operatorname{Ran}(T)$. (ii) By the rank-nullity theorem, $\dim C(A) + \dim N(A) = n$. Since $\dim N(A) \geq 0$, this means $\dim C(A) \leq n$. So, if $C(A) = \mathbb{R}^m$, so $\dim C(A) = m$, we conclude that $m \leq n$. (iii) If $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for every $\underset{\sim}{b} \in \mathbb{R}^m$ then $N(A) = \{0\}$; otherwise any non-zero element (as well as 0) would solve $A\underset{\sim}{x} = 0$. Thus, by the rank-nullity theorem, $\dim C(A) = n$ as desired.

3(b) (3 points): (i) Find the matrices A_1, A_2 of the orthogonal projections P_{V_j}, $j = 1, 2$, to $V_1 = \operatorname{Span}\{(1, 1, 1)^{\mathrm{T}}\}$ and $V_2 = \operatorname{Span}\{(1, -1, 0)^{\mathrm{T}}\}$ in $\mathbb{R}^3$. (ii) Show that the matrix of the orthogonal projection P_V to $V = \operatorname{Span}\{(1, 1, 1)^{\mathrm{T}}, (1, -1, 0)^{\mathrm{T}})\}$ is $A_1 + A_2$.
Hint for (ii): Note that $(1, 1, 1)^{\mathrm{T}}$ and $(1, -1, 0)^{\mathrm{T}}$ are orthogonal.

Solution: The orthogonal projection of a vector $\underset{\sim}{x}$ to the span of a non-zero vector $\underset{\sim}{v}$ is $P_{\operatorname{span} \underset{\sim}{v}} \underset{\sim}{x} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{x}}{\|\underset{\sim}{v}\|^2} \underset{\sim}{v}$, i.e. explicitly the ith coordinate of $P_{\operatorname{span} \underset{\sim}{v}} \underset{\sim}{e}_j$ is $P_{\operatorname{span} \underset{\sim}{v}} \underset{\sim}{e}_j = \frac{\underset{\sim}{v} \cdot \underset{\sim}{e}_j}{\|\underset{\sim}{v}\|^2} v_i = \frac{v_j}{\|\underset{\sim}{v}\|^2} v_i$, which says exactly that the ijth entry of the matrix of the projection is $\frac{v_i v_j}{\|\underset{\sim}{v}\|^2}$. Concretely, this gives

$$A_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We claim that the orthogonal projection to $V = \operatorname{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ is $P_{\operatorname{span} \underset{\sim}{v}_1} + P_{\operatorname{span} \underset{\sim}{v}_2}$. Indeed, suppose $\underset{\sim}{x} \in \mathbb{R}^3$; we know that $\underset{\sim}{x} = \underset{\sim}{x}^{\parallel} + \underset{\sim}{x}^{\perp}$ with $x^{\parallel} \in V$ and $x^{\perp} \in V^{\perp}$, the decomposition of $\underset{\sim}{x}$ is unique, and $P_V \underset{\sim}{x} = \underset{\sim}{x}^{\parallel}$. Now, note that $P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} + P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x} \in \operatorname{Span}\{\underset{\sim}{v}_1\} + \operatorname{Span}\{\underset{\sim}{v}_2\} = \operatorname{Span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\} = V$, so it suffices to show that $\underset{\sim}{x} - (P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} + P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x}) \in V^{\perp}$. But by Problem 2(a)/the lecture/book, this is equivalent to asking that $(\underset{\sim}{x} - (P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} + P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x})) \cdot \underset{\sim}{v}_j = 0$, $j = 1, 2$. We consider $j = 1$; $j = 2$ is similar. Then $(\underset{\sim}{x} - P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x}) \cdot \underset{\sim}{v}_1 = 0$ because $P_{\operatorname{span} \underset{\sim}{v}_1}$ is orthogonal projection to $\operatorname{span}\{v_1\}$ so $\underset{\sim}{x} - P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} \in \operatorname{span}\{v_1\}^{\perp}$. On the other hand, $P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x} \cdot \underset{\sim}{v}_1 = 0$ since $P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x} \in \operatorname{span}\{\underset{\sim}{v}_2\}$, and $\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_1 = 0$. In summary, $(\underset{\sim}{x} - (P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} + P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x})) \cdot \underset{\sim}{v}_j = 0$, $j = 1, 2$, proving that $\underset{\sim}{x} - (P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} + P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x}) \in V^{\perp}$, so $P_V \underset{\sim}{x} = P_{\operatorname{span} \underset{\sim}{v}_1} \underset{\sim}{x} + P_{\operatorname{span} \underset{\sim}{v}_2} \underset{\sim}{x}$. Correspondingly, the matrix of the orthogonal projection to V is

$$A = A_1 + A_2 = \begin{pmatrix} \frac{5}{6} & \frac{-1}{6} & \frac{1}{3} \\ \frac{-1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

4 (6 points): Find (i) $\operatorname{rref} A$ (showing all row operations), (ii) a basis for the null space $N(A)$, (iii) a basis for the column space of A and (iv) $\dim N(A^{\mathrm{T}})$, if

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ -1 & -2 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Solution: (i)

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ -1 & -2 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix} r_2 \mapsto r_2 + r_1 \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix} r_2 \leftrightarrow r_3 \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$$r_1 \mapsto r_1 - 3r_3 \begin{pmatrix} 1 & 2 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix} r_1 \mapsto r_1 - 2r_2 \begin{pmatrix} 1 & 0 & 0 & -2 & -9 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

(ii)

$$\begin{aligned} \operatorname{rref} A\underline{x} = \underline{0} &\iff \big(x_1 = 2x_4 + 9x_5, x_2 = -2x_5, x_3 = -x_4 - 2x_5\big) \\ &\iff \underline{x} = x_4(2,0,-1,1,0)^{\mathrm{T}} + x_5(9,-2,-2,0,1)^{\mathrm{T}} \end{aligned}$$

with x_4, x_5 arbitrary, so $N(A) = N(\operatorname{rref} A) = \operatorname{span}\{(2,0,-1,1,0)^{\mathrm{T}}, (9,-2,-2,0,1)^{\mathrm{T}}\}$, and these two vectors are indeed linearly independent (by inspection of the last two components, or the general result from lecture), so they give a basis for $N(A)$.

(iii) The pivot columns of $\operatorname{rref} A$ are the first, second and third columns, so from lecture a basis for $C(A)$ is obtained by taking the first, second and third columns of A; that is, a basis for $C(A)$ is $(1,-1,0)^{\mathrm{T}}, (2,-2,1)^{\mathrm{T}}, (3,-2,0)^{\mathrm{T}}$.

(iv) $N(A^{\mathrm{T}})$, which is a subspace of $\mathbb{R}^3$, is the orthocomplement of $C(A)$, and the sum of the dimensions of a subspace and its orthocomplement is that of the total space. Since $\dim C(A) = 3$, $\dim N(A^{\mathrm{T}}) = \dim \mathbb{R}^3 - \dim C(A) = 0$, and thus $N(A^{\mathrm{T}}) = \{0\}$.

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