Mathematics Department Stanford University
Math 51H Mid-Term 1

October 14, 2014

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ___________________________

I understand and accept the provisions of the honor code (Signed) ___________________________

1 (a) (3 points): (i) Give the ε, N definition of "$\lim a_n = \ell$," where $\{a_n\}_{n=1,2,\ldots}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and (ii) use your definition to prove that if $\{a_n\}_{n=1,2,\ldots}, \{b_n\}_{n=1,2,\ldots}$ satisfy $\lim a_n = \ell$, $\lim b_n = m$, then $\lim(a_n - b_n) = \ell - m$.

Note for (ii): You may not use any of our limit theorems to prove (ii), only the definition of the limit, and properties of the reals.

1(b) (3 points): Suppose that S is a bounded non-empty subset of $\mathbb{R}$ with the property that $x, y \in S$, $x < z < y$ imply that $z \in S$. Let $a = \inf S$, $b = \sup S$. Show that S must be one of the intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$ (with only the last possibility if $a = b$).

Hint for (b): The conclusion is equivalent to $a < z < b$ implying that $z \in S$ together with $z \notin [a, b]$ implying $z \notin S$.

2(a) (3 points): (i) Give the definition of a collection $\underline{v}_1, \ldots, \underline{v}_k$ of vectors in $\mathbb{R}^n$ being linearly independent, and (ii) if $\underline{v}_1, \ldots, \underline{v}_k$ are non-zero mutually orthogonal (i.e. $\underline{v}_i \cdot \underline{v}_j = 0 \, \forall \, i \neq j$) vectors in $\mathbb{R}^n$, prove that $\underline{v}_1, \ldots, \underline{v}_k$ are linearly independent.

2(b) (4 points): Suppose that V is a non-trivial subspace of $\mathbb{R}^n$. Show that there is an orthogonal basis of V, i.e. that there is a basis $\{\underline{v}_1, \ldots, \underline{v}_k\}$ for V with $\underline{v}_i \cdot \underline{v}_j = 0$ if $i \neq j$. (You may assume the result of part (a) even if you have not proved it.)

Hint for (b): As in the proof of the basis theorem, consider a maximum size set of non-zero mutually orthogonal vectors; you need to show along the way that this exists. Orthocomplements may be useful in proving the spanning property.

3(a) (3 points): Suppose A is an $n \times n$ matrix and $\underset{\sim}{b} \in \mathbb{R}^n$. Show that if $C(A) = \mathbb{R}^n$ then that $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for each $\underset{\sim}{b} \in \mathbb{R}^n$. (You need to show both existence and uniqueness.)
Hint: Use the rank/nullity theorem.

3(b) (3 points): Suppose that V, W are subspaces of $\mathbb{R}^n$ and $V \subset W$. Show that if $\dim V = \dim W$ then $V = W$.

4 (6 points): Find (i) rref A (showing all row operations), (ii) a basis for the null space $N(A)$ and (iii) a basis for the column space of A, if

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ -1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

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