Mathematics Department Stanford University
Math 51H Mid-Term 1

October 14, 2014

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
T/25	

Name (Print Clearly): ________________________

I understand and accept the provisions of the honor code (Signed) ________________________

1 (a) (3 points): (i) Give the ε, N definition of "$\lim a_n = \ell$," where $\{a_n\}_{n=1,2,\ldots}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and (ii) use your definition to prove that if $\{a_n\}_{n=1,2,\ldots}, \{b_n\}_{n=1,2,\ldots}$ satisfy $\lim a_n = \ell$, $\lim b_n = m$, then $\lim(a_n - b_n) = \ell - m$.

Note for (ii): You may not use any of our limit theorems to prove (ii), only the definition of the limit, and properties of the reals.

Solution: $\lim a_n = \ell$ means that for each $\varepsilon > 0$ there is N such that $|a_n - \ell| < \varepsilon$ for all $n \geq N$.

Now, suppose $\varepsilon > 0$. Since $\lim a_n = \ell$, applying the definition with $\varepsilon/2 > 0$ means that there is N_1 such that $n \geq N_1$ implies $|a_n - \ell| < \varepsilon/2$. Similarly, $\lim b_n = m$ means that there is N_2 such that $|b_n - m| < \varepsilon/2$ for $n \geq N_2$. Thus, for $n \geq N = \max(N_1, N_2)$,

$$|(a_n - b_n) - (\ell - m)| = |(a_n - \ell) + (m - b_n)| \leq |a_n - \ell| + |m - b_n| < \varepsilon,$$

proving the conclusion.

1(b) (3 points): Suppose that S is a bounded non-empty subset of $\mathbb{R}$ with the property that $x, y \in S$, $x < z < y$ imply that $z \in S$. Let $a = \inf S$, $b = \sup S$. Show that S must be one of the intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$ (with only the last possibility if $a = b$).

Hint for (b): The conclusion is equivalent to $a < z < b$ implying that $z \in S$, together with $z \notin [a, b]$ implying $z \notin S$.

Solution: Suppose $a < z < b$. As $z > a$, z is not a lower bound for S, i.e. there exists some $x \in S$ such that $x < z$. Similarly, as $z < b$, z is not an upper bound for S so there exists some $y \in S$ such that $z < y$. Thus, $x < z < y$, $x, y \in S$, so $z \in S$. Thus, $(a, b) \subset S$.

Now, if $z < a$ then $z \notin S$ since a is a lower bound for S, and similarly if $z > b$ then $z \notin S$ since b is an upper bound for S. Thus $(-\infty, a) \cup (b, +\infty) \subset S^c = \mathbb{R} \setminus S$.

As $\mathbb{R} = (-\infty, a) \cup \{a\} \cup (a, b) \cup \{b\} \cup (b, \infty)$, the only question is whether a and b are in S; listing the four possibilities (only two if $a = b$) gives the four intervals (only one if $a = b$ as we assumed that S was non-empty).

2(a) (3 points): (i) Give the definition of a collection $\underline{v}_1, \dots, \underline{v}_k$ of vectors in $\mathbb{R}^n$ being linearly independent, and (ii) if $\underline{v}_1, \dots, \underline{v}_k$ are non-zero mutually orthogonal (i.e. $\underline{v}_i \cdot \underline{v}_j = 0 \, \forall \, i \neq j$) vectors in $\mathbb{R}^n$, prove that $\underline{v}_1, \dots, \underline{v}_k$ are linearly independent.

Solution: A collection $\underline{v}_1, \dots, \underline{v}_k$ of vectors in $\mathbb{R}^n$ is linearly independent if there is no non-trivial linear combination of them which is 0, i.e. $\sum_{j=1}^k c_j \underline{v}_j = 0$ implies $c_j = 0$ for all j. (ii) $\sum_{j=1}^k c_j \underline{v}_j = \underline{0} \Rightarrow 0 = \underline{v}_i \cdot (\sum_{j=1}^k c_j \underline{v}_j) \Rightarrow 0 = \sum_{j=1}^k c_j v_i \cdot v_j = c_i \|v_i\|^2 \Rightarrow c_i = 0$ for each $i = 1, \dots, k$, where we used the fact that $v_i \cdot v_j = 0$ if $j \neq i$ and $= \|\underline{v}_i\|^2 \neq 0$ if $j = i$.

2(b) (4 points): Suppose that V is a non-trivial subspace of $\mathbb{R}^n$. Show that there is an orthogonal basis of V, i.e. that there is a basis $\{\underline{v}_1, \dots, \underline{v}_k\}$ for V with $\underline{v}_i \cdot \underline{v}_j = 0$ if $i \neq j$. (You may assume the result of part (a) even if you have not proved it.)

Hint for (b): As in the proof of the basis theorem, consider a maximum size set of non-zero mutually orthogonal vectors; you need to show along the way that this exists. Orthocomplements may be useful in proving the spanning property.

Solution: If $\{\underline{v}_1, \dots, \underline{v}_k\}$ is any collection of mutually orthogonal non-zero vectors, it is linearly independent by part (a), so by the linear dependence lemma, $k \leq n$. Moreover, as V is non-trivial, there exists a non-zero vector $\underline{v}$ in it; then $\{\underline{v}\}$ is an orthogonal collection (a set with one non-zero element). Now let

$$S = \{k : \exists \{\underline{v}_1, \dots, \underline{v}_k\} \text{ mutually orthogonal non-zero in } V\}.$$

Then S is a non-empty set of positive integers, bounded above by n, thus it has a maximal element; let $k = \max S$. Let $\{\underline{v}_1, \dots, \underline{v}_k\}$ be mutually orthogonal in V; these exist by the very definition of k. Now let $W = \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\} + V^\perp$, with $V^\perp$ the orthocomplement of V in $\mathbb{R}^n$. Then $W \oplus W^\perp = \mathbb{R}^n$. If $W^\perp \neq \{0\}$ then there exists $\underline{x} \neq 0$, $\underline{x} \in W^\perp$, so $\underline{x}$ is orthogonal to all $\underline{v}_j$, and $\underline{x}$ is orthogonal to all elements of $V^\perp$. The latter means $\underline{x} \in (V^\perp)^\perp = V$. Then $\{\underline{v}_1, \dots, \underline{v}_k, \underline{x}\}$ are $k+1$ mutually orthogonal non-zero vectors in V, so $k+1 \in S$, which contradicts our choice of k. Thus, $W^\perp = \{0\}$, so $W = \mathbb{R}^n$, i.e. $\operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\} + V^\perp = \mathbb{R}^n$. Notice that the first summand is a subspace of V, so any $\underline{v} \in V$ can be written as $\underline{y} + \underline{z}$, $\underline{y} \in \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\} \subset V$, $\underline{z} \in V^\perp$. Since there is a unique way of writing any element of $\mathbb{R}^n$, in particular any element $\underline{v}$ of V as a vector in V plus one in $V^\perp$, and as $\underline{v} = \underline{v} + 0$ is such a decomposition, we conclude that $\underline{v} = \underline{y} \in \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\}$, so $V = \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\}$. Since $\{\underline{v}_1, \dots, \underline{v}_k\}$ are linearly independent, this means that they form a basis of V, as claimed.

Alternative (simpler) argument using orthocomplements within V: Find $\underline{v}_1, \dots, \underline{v}_k$ as above, but let $W = \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\}$. Now, $V = W \oplus W_V^\perp$ with $W_V^\perp = \{\underline{v} \in V : \underline{v} \cdot \underline{w} = 0 \; \forall \; \underline{w} \in W\}$ the orthocomplement of W in V, so $W = V$ if and only if $W_V^\perp = \{0\}$. But if $W_V^\perp \neq \{0\}$ then there is $\underline{x} \in W$ such that $x \neq 0$, and then $\{\underline{v}_1, \dots, \underline{v}_k, \underline{x}\}$ are $k+1$ mutually orthogonal non-zero vectors in V, so $k+1 \in S$, which contradicts our choice of k. So $W_V^\perp = 0$, and thus $V = \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_k\}$. Since $\{\underline{v}_1, \dots, \underline{v}_k\}$ are linearly independent, this means that they form a basis of V, as claimed.

3(a) (3 points): Suppose A is an $n \times n$ matrix and $\underset{\sim}{b} \in \mathbb{R}^n$. Show that if $C(A) = \mathbb{R}^n$ then that $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution for each $\underset{\sim}{b} \in \mathbb{R}^n$. (You need to show both existence and uniqueness.)
Hint: Use the rank/nullity theorem.

Solution: For any $\underset{\sim}{x} \in \mathbb{R}^n$, $A\underset{\sim}{x} = \sum_{j=1}^n x_j \underset{\sim}{\alpha}_j$, where $\underset{\sim}{\alpha}_j$ is the j'th column of A. Thus $\{A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n\} = \text{span}\{\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_n\} = C(A)$, so $\underset{\sim}{b} \in \{A\underset{\sim}{x} : \underset{\sim}{x} \in \mathbb{R}^n\} \iff \underset{\sim}{b} \in C(A)$. Correspondingly, if $C(A) = \mathbb{R}^n$, then $A\underset{\sim}{x} = \underset{\sim}{b}$ has a solution for all $\underset{\sim}{b} \in \mathbb{R}^n$.

If $C(A) = \mathbb{R}^n$ then the rank/nullity theorem tells us that the dimension of $N(A)$ is 0, i.e. $N(A) = \{0\}$. But, if $A\underset{\sim}{x} = \underset{\sim}{b}$ and $A\underset{\sim}{y} = \underset{\sim}{b}$ then $A(\underset{\sim}{x} - \underset{\sim}{y}) = A\underset{\sim}{x} - A\underset{\sim}{y} = \underset{\sim}{b} - \underset{\sim}{b} = 0$, so $\underset{\sim}{x} - \underset{\sim}{y} \in N(A)$. So $N(A) = \{0\}$ gives $\underset{\sim}{x} = \underset{\sim}{y}$, which is the desired uniqueness.

3(b) (3 points): Suppose that V, W are subspaces of $\mathbb{R}^n$ and $V \subset W$. Show that if $\dim V = \dim W$ then $V = W$.

Solution: If V is the trivial subspace of $\mathbb{R}^n$, then $\dim W = 0$ and thus W is also the trivial subspace, completing the proof in this case.

If V is not the trivial subspace, then V has a basis $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k\}$. These are linearly independent and lie in W, thus by the basis theorem, applied to W, there exists a basis $\{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_k, \ldots, \underset{\sim}{w}_m\}$ of W with $m \geq k$ and with $\underset{\sim}{w}_j = \underset{\sim}{v}_j$ for $j \leq k$. In particular, $\dim W = m \geq k = \dim V$. Since we know $\dim V = \dim W$, we have $m = k$, i.e. $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k\}$ is a basis of W as well. Thus, W is the span of these vectors, i.e. $W = V$.

4 (6 points): Find (i) rref A (showing all row operations), (ii) a basis for the null space $N(A)$ and (iii) a basis for the column space of A, if

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ -1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution: (i)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ -1 & 0 & 0 & 1 & 2 \end{pmatrix}_{r_3 \mapsto r_3 + r_1} \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 1 & 0 & 1 & 5 \end{pmatrix} r_2 \mapsto r_2/2 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1 & 5 \end{pmatrix}$$

$$r_3 \mapsto r_3 - r_2 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & -2 & 1/2 & 11/2 \end{pmatrix} r_3 \mapsto -r_3/2 \begin{pmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/4 & -11/4 \end{pmatrix}$$

$$r_1 \mapsto r_1 - r_2 \begin{pmatrix} 1 & 0 & -2 & -1/2 & 7/2 \\ 0 & 1 & 2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/4 & -11/4 \end{pmatrix} \begin{matrix} r_1 \mapsto r_1 + 2r_3 \\ r_2 \mapsto r_2 - 2r_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1/4 & -11/4 \end{pmatrix}$$

(ii)

$$\begin{aligned} \operatorname{rref} A\underline{x} = \underline{0} &\iff \left(x_1 = x_4 + 2x_5, x_2 = -x_4 - 5x_5, x_3 = \frac{1}{4}x_4 + \frac{11}{4}x_5\right) \\ &\iff \underline{x} = x_4(1, -1, \frac{1}{4}, 1, 0)^{\mathrm{T}} + x_5(2, -5, \frac{11}{4}, 0, 1)^{\mathrm{T}} \end{aligned}$$

with x_4, x_5 arbitrary, so $N(A) = N(\operatorname{rref} A) = \operatorname{span}\{(1, -1, \frac{1}{4}, 1, 0)^{\mathrm{T}}, (2, -5, \frac{11}{4}, 0, 1)^{\mathrm{T}}\}$, and these two vectors are indeed linearly independent (by inspection of the last two components, or the general result from lecture), so they give a basis for $N(A)$.

(iii) The pivot columns of rref A are the first, second and third columns, so from lecture a basis for $C(A)$ is obtained by taking the first, second and third columns of A; that is, a basis for $C(A)$ is $(1, 0, -1)^{\mathrm{T}}, (1, 2, 0)^{\mathrm{T}}, (0, 4, 0)^{\mathrm{T}}$.

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