Mathematics Department Stanford University
Math 51H Mid-Term 1

October 15, 2013

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Name (Print Clearly): ______________________

I understand and accept the provisions of the honor code (Signed) ______________________

1 (a) (3 points): Give the definition of "$\lim a_n = \ell$," where $\{a_n\}_{n=1,2,...}$ is a given sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$, and use your definition to prove $\ell \geq 0$, assuming that the limit ℓ exists and that $a_n \geq 0 \, \forall n$.

Solution: $\lim a_n = \ell$ means that for each $\varepsilon > 0$ there is N such that $|a_n - \ell| < \varepsilon$ for all $n \geq N$. This says $\ell - \varepsilon < a_n < \ell + \varepsilon$ for all $n \geq N$. Now if $\ell < 0$ then we can take $\varepsilon = -\ell$, in which case the above implies $\exists N$ such that $a_n < \ell - \ell = 0$ for all $n \geq N$, contradicting the fact that $a_n \geq 0 \, \forall n$.

(b) (3 points): Suppose that S is a non-empty subset of $\mathbb{R}$ which is bounded above, and let $\alpha = \sup S$.

(i) Prove that for each $\varepsilon > 0$ there is $x \in S$ with $x > \alpha - \varepsilon$.

(ii) Prove that there is a sequence $\{x_n\}_{n=1,2,...}$ with $x_n \in S$ for each n and $\lim x_n = \alpha$.

Solution (i): If this fails for any $\varepsilon > 0$, then $\alpha - \varepsilon$ would be an upper bound for S, contradicting the fact that α is the least upper bound.

Solution (ii): For each $n = 1, 2, \ldots$ we can use (i) with $\varepsilon = 1/n$, thus showing that there is $x_n \in S$ with $x_n > \alpha - 1/n$. Then $\alpha - 1/n \leq x_n \leq \alpha$ and so the Sandwich Theorem gives $\lim x_n = \alpha$.

2 (a) (3 points): Suppose $\underline{a}\,,\underline{b}$ are distinct vectors in $\mathbb{R}^n$.

(i) Give the definition of "the line ℓ through $\underline{a}$ parallel to $\underline{b}-\underline{a}$," and find the vector $\underline{v}\in\ell$ which is equi-distant from $\underline{a}\,,\underline{b}$ (i.e. $\|\underline{v}-\underline{a}\|=\|\underline{v}-\underline{b}\|$).

(ii) If $\underline{v}$ is as in (i) and $\|\underline{a}\|=\|\underline{b}\|$, prove $\underline{v}\cdot(\underline{b}-\underline{a})=0$.

Solution (i): The line ℓ through $\underline{a}$ parallel to $\underline{b}-\underline{a}$ is defined by $\ell=\{\underline{a}+t(\underline{b}-\underline{a}):t\in\mathbb{R}\}$. We want the mid-point of the part of the line joining $\underline{a}$ to $\underline{b}$ and this intuitively should be given by taking $t=\frac{1}{2}$, i.e. $\underline{v}=\underline{a}+\frac{1}{2}(\underline{b}-\underline{a})=\frac{1}{2}(\underline{a}+\underline{b})$. To check that this works, we calculate $\underline{v}-\underline{a}=\frac{1}{2}(\underline{b}-\underline{a})$, whereas $\underline{v}-\underline{b}=\frac{1}{2}(\underline{a}-\underline{b})=-\frac{1}{2}(\underline{b}-\underline{a})$, so indeed $\|\underline{v}-\underline{a}\|=\|\underline{v}-\underline{b}\|$.

Solution (ii): $\underline{v}\cdot(\underline{b}-\underline{a})=\frac{1}{2}(\underline{b}+\underline{a})\cdot(\underline{b}-\underline{a})=\frac{1}{2}(\underline{b}\cdot\underline{b}-\underline{a}\cdot\underline{a}+\underline{a}\cdot\underline{b}-\underline{b}\cdot\underline{a})=\frac{1}{2}(\|\underline{b}\|^2-\|\underline{a}\|^2)=0$.

(b) (3 points): Prove that $2\,|\underline{x}\cdot\underline{y}|\,\|\underline{x}\|^2\le\|\underline{x}\|^6+\|\underline{y}\|^2$ for all vectors $\underline{x}\,,\underline{y}\in\mathbb{R}^n$.

Solution: The Cauchy-Schwarz inequality says $|\underline{x}\cdot\underline{y}|\le\|\underline{x}\|\|\underline{y}\|$, so $\|\underline{x}\|^6+\|\underline{y}\|^2-2\,|\underline{x}\cdot\underline{y}|\,\|\underline{x}\|^2\ge\|\underline{x}\|^6+\|\underline{y}\|^2-2\,\|\underline{x}\|^3\|y\|=(\|\underline{x}\|^3-\|\underline{y}\|)^2\ge0$.

3 (a) (4 points): Suppose

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & 0 & 2 \end{pmatrix}$$

Find (i) a basis for the null space $N(A)$ of A (show all row operations!), and (ii) a basis for the column space $C(A)$.

Make sure you justify your results by referring to the appropriate results from lecture.

Solution: We compute the reduced row echelon form of A as follows:

$$r_3 \leftrightarrow r_4 \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{matrix} r_2 \mapsto r_2 - 2r_1 \\ r_3 \mapsto r_3 - r_1 \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} r_3 \mapsto r_3 - r_2 \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\begin{matrix} r_3 \mapsto r_3/2 \\ r_4 \mapsto r_4 - r_3/2 \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} r_1 \mapsto r_1 - r_3 \\ r_2 \mapsto r_2 + r_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\underline{x}$ is a solution of $A\underline{x} = \underline{0} \iff x_3 = x_4 - x_5,\ x_2 = 0,\ x_1 = -2x_4 \iff \underline{x} = (-2x_4, 0, x_4 - x_5, x_4, x_5)^{\mathrm{T}} = x_4(-2,0,1,1,0)^{\mathrm{T}} + x_5(0,0,-1,0,1)^{\mathrm{T}}$, where x_4, x_5 are arbitrary reals, so the null space is the subspace spanned by $(-2,0,1,1,0)^{\mathrm{T}}$ and $(0,0,-1,0,1)^{\mathrm{T}}$. Since $(-2,0,1,1,0)^{\mathrm{T}}$ and $(0,0,-1,0,1)^{\mathrm{T}}$ are l.i. (which can be justified either by a direct check or by the fact that we are following the general method of lecture, which was shown always to yield l.i. vectors and hence a basis for the null space), this is a 2 dimensional space and $(-2,0,1,1,0)^{\mathrm{T}}$ and $(0,0,-1,0,1)^{\mathrm{T}}$ are a basis.

(ii) In lecture we proved that if $j_1, \ldots, j_Q$ are the column numbers of the pivot columns of $\mathrm{rref}\,(A)$ then the columns $\underline{\alpha}_{j_1}, \ldots, \underline{\alpha}_{j_Q}$ of A are a basis for $C(A)$. In this case we have $Q = 3$ and $j_1, j_2, j_3 = 1, 2, 3$ respectively, so the first 3 cols. of A are a basis for $C(A)$.

3 (b) (3 points): Suppose $V \subset \mathbb{R}^n$ is a non-trivial subspace of dimension k. Give the proof that any k vectors $\underline{v}_1, \ldots, \underline{v}_k \in V$ which span V (i.e. $V = \mathrm{span}\{\underline{v}_1, \ldots, \underline{v}_k\}$) must automatically be a basis for V.

Solution: Since $\underline{v}_1, \ldots, \underline{v}_k$ are given to span V, we just have to show they are l.i. Suppose on the contrary that they are l.d. Then from lecture at least one of them, say $\underline{v}_j$, is a linear combination of the others. Thus $\underline{v}_j = \sum_{i \neq j} c_i \underline{v}_i$ for some constants c_i, $i \neq j$. But then any linear combination of $\underline{v}_1, \ldots, \underline{v}_k$ can be rewritten as a linear combination of $\underline{v}_i$, $i \neq j$. Then $V = \mathrm{span}\{\underline{v}_1, \ldots, \underline{v}_k\} = \mathrm{span}\{v_i : i \neq j\}$. But then a basis $\underline{w}_1, \ldots, \underline{w}_k$ for V would consist of k l.i. vectors in the span of the $k-1$ vectors $\underline{v}_i$, $i \neq j$, contradicting the linear dependence lemma.

4 (a) (3 points): Suppose A is an $m \times n$ matrix and $\underline{b} \in \mathbb{R}^m$. (i) Give the proof that $A\underline{x} = \underline{b}$ has at least one solution $\underline{x} \in \mathbb{R}^n \iff \underline{b} \in C(A)$, and (ii) In case $m = n$ and $N(A) = \{\underline{0}\}$, prove that $A\underline{x} = \underline{b}$ has a solution for each $\underline{b} \in \mathbb{R}^n$.

Hint for (ii): Use the rank/nullity theorem.

Solution: (i) As we checked in lecture, $A\underline{x} = \sum_{j=1}^n x_j \underline{\alpha}_j$, where $\underline{\alpha}_j$ is the j'th column of A, so $\exists\, \underline{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ with $A\underline{x} = \underline{b} \iff \sum_{j=1}^n x_j \underline{\alpha}_j = \underline{b}$. Thus there is a solution of $A\underline{x} = \underline{b}$ if and only if some linear combination of the $\underline{\alpha}_j$ is equal to $\underline{b}$, i.e. if and only if $\underline{b} \in \operatorname{span}\{\underline{\alpha}_1, \ldots, \underline{\alpha}_n\} = C(A)$.

(ii) If $N(A) = \{\underline{0}\}$ then the rank/nullity theorem tells us that the dimension of $C(A) = n$. That is $C(A)$ is a subspace of $\mathbb{R}^n$ of dimension n and hence it must be all of $\mathbb{R}^n$ because by a theorem of lecture any k l.i. vectors in a k-dimensional subspace of $\mathbb{R}^n$ must be a basis for that subspace. Thus $C(A) = \mathbb{R}^n$ and hence by part (i) $A\underline{x} = \underline{b}$ has a solution for all $\underline{b} \in \mathbb{R}^n$.

4 (b) (2 points): If V is a subspace of $\mathbb{R}^n$, give the definition of $V^\perp$. Prove (i) that $V^\perp$ is a subspace, and (ii) that $V \cap V^\perp = \{\underline{0}\}$.

Solution: $V^\perp$ is the set of all vectors $\underline{y} \in \mathbb{R}^n$ such that $\underline{y} \cdot \underline{v} = 0$ for every $\underline{v} \in V$.

(i) First note that (a) trivially $\underline{0} \in V^\perp$, and (b) $\underline{x}, \underline{y} \in V^\perp \Rightarrow \underline{v} \cdot (\underline{x} + \underline{y}) = \underline{v} \cdot \underline{x} + \underline{v} \cdot \underline{y} = 0 + 0 = 0$ for each $\underline{v} \in V$, so $\underline{x} + \underline{y} \in V^\perp$. Finally (c) $\lambda \in \mathbb{R}$ and $\underline{y} \in V^\perp \Rightarrow (\lambda \underline{y}) \cdot \underline{v} = \lambda(\underline{y} \cdot \underline{v}) = \lambda.0 = 0$ for each $\underline{v} \in V$, so $\lambda \underline{y} \in V^\perp$. Thus $V^\perp$ has the required 3 properties, hence is a subspace.

(ii) $\underline{w} \in V \cap V^\perp \Rightarrow \underline{w} \in V^\perp \Rightarrow \underline{w} \cdot \underline{v} = 0\, \forall \underline{v} \in V$. But $\underline{w} \in V$, so then $\underline{w} \cdot \underline{w} = \|\underline{w}\|^2 = 0$, so $\underline{w} = \underline{0}$.