Mathematics Department Stanford University
Math 51H Final Examination, December 7, 2015

2 Hours

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Question 7 is extra credit only! Work on it only if you are done with the other problems!

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Q.6	
T/35	
Q.7	

Name (Print Clearly): ____________________

I understand and accept the provisions of the honor code (Signed) ____________________

1(a) (3 points): Find (with detailed proof!) $\det A$, $\det B$ and $\det(AB)$ if

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) (3 points) Suppose $A : V \to W$ is linear where V, W are finite dimensional real vector spaces. Let $N(A) = \{x \in V : \ Ax = 0\}$ and $R(A) = \{Ax : \ x \in V\} \subset W$. Show that $\dim N(A) + \dim R(A) = \dim V$.

Note: If you want, you may use matrices, but be specific about the correspondence between matrices and operators. Also, this problem works over any field.

2 (a) ($2\frac{1}{2}$ points): Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and let a be a given point of $\mathbb{R}^n$. Show that if there is $\rho > 0$ such that the partial derivatives $D_j f(x), j = 1, \ldots, n$ exist for $\|x - a\| < \rho$ and are continuous at a, then f is differentiable at a.

(b) ($3\frac{1}{2}$ points): State (without proof) the Lagrange multiplier theorem, and use it (together with any other theorems from lecture that you need) to find a point where the function $xy + z^3$ takes its maximum subject to the constraint that $x^4 + y^4 + z^4 = 1$, and justify your answer.

Note: Your discussion should include the reason that the maximum exists.

3(a) (3 points): Suppose (X, d_X) and (Y, d_Y) are metric spaces (if you wish, you may assume $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ with the relative metric). Show that $f : X \to Y$ is continuous if and only if for all $U \subset Y$ open, $f^{-1}(U) = \{x \in X : \ f(x) \in U\}$ is open.

(b) (3 points) Assume $\sin x, \cos x$ are defined as usual by the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ respectively. Then (i) prove $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$, and (ii) prove the identity $\sin(x+a) = \sin x \cos a + \cos x \sin a$.

Hint for (ii): For fixed a define $f_a(x) = \sin(x+a) - \sin x \cos a - \cos x \sin a$ and start by showing that $\frac{d^n}{dx^n} f_a(x)|_{x=0} = 0$ for all $n = 0, 1, 2, \ldots$.

4(a) ($3\frac{1}{2}$ points): Find all eigenvalues and corresponding eigenvectors for the matrix

$$\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

(b) ($1\frac{1}{2}$ points): Prove that the quadratic form $Q(h) = 5h_1^2 - 4h_1h_2 + 2h_2^2 + 2h_1h_3 + 4h_2h_3 + 6h_3^2$ is positive definite.

Hint: compare Q with the quadratic form of the matrix in part (a).

5(a) (3 points): Suppose that $\{M_n\}_{n=1}^{\infty}$ is a sequence with $M_n \geq 0$ for all n, and $\sum_{n=1}^{\infty} M_n$ converges. Show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in a complete normed vector space $(V, \|.\|)$ (if you wish, you may take $V = \mathbb{R}^m$ with the standard norm) and $\|x_n\| \leq M_n$ for all n, then $\sum_{n=1}^{\infty} x_n$ converges in V, i.e. $\lim_{k\to\infty} \sum_{n=1}^{k} x_n$ exists. (This is the Weierstrass M-test.)

(b) (3 points): Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^4} \cos(nx)$ converges uniformly to a C^1 function $f(x)$ on $[0, 2\pi]$ and $f'(x) = -\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$.

6(a) (3 points): Suppose V is a finite dimensional real vector space, and $e_1, \ldots, e_n$ is a basis for V. Show that the linear maps $f_i : V \to \mathbb{R}$, $i = 1, \ldots, n$, defined by $f_i(\sum_{j=1}^n a_j e_j) = a_i$ give a basis (called the dual basis) of $V^* = \mathcal{L}(V, \mathbb{R})$.

(b) (3 points): Suppose that $A \in \mathcal{L}(V, V)$ is linear, V, etc., as above. Show that $\operatorname{trace} A = \sum_{j=1}^n f_j(Ae_j)$ is independent of the choice of the basis of V, and if A is symmetric, then $\operatorname{trace} A$ is the sum of the eigenvalues of A, counted with multiplicity, i.e. if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues, then $\operatorname{trace} A = \sum_{j=1}^k \lambda_j \dim N(A - \lambda_j I)$.

Note: $f_j(Ae_j)$ is the jj entry of the matrix of A in the basis $e_1, \ldots, e_n$, so the trace is the sum of the diagonal entries of the matrix.

7 (6 points extra credit only): Suppose $a_{mn} \geq 0$ for $m, n \geq 1$ integer. Show that the set $\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\}$ is bounded above if and only if for each m, $\sum_{n=1}^{\infty} a_{mn}$ converges and $\{\sum_{m=1}^{M} \sum_{n=1}^{\infty} a_{mn} :\ M \geq 1\}$ is bounded above. Show moreover that in this case

$$\sup\left\{ \sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\right\} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn},$$

where both sums on the right hand side converge, and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}.$$

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