Mathematics Department Stanford University
Math 51H Final Examination, December 7, 2015

Solutions

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Question 7 is extra credit only! Work on it only if you are done with the other problems!

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Q.6	
T/35	
Q.7	

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________

1(a) (3 points): Find (with detailed proof!) $\det A$, $\det B$ and $\det(AB)$ if

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution: The determinant of a matrix C with entries c_{ij} is $\det C = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) c_{\sigma_1 1} \dots c_{\sigma_n n}$, where the sum is over permutations of $\{1, \dots, n\}$. This means that each summand takes one entry from each column of C, and the entries corresponding to different columns must come from different rows. In particular, in the case of A there are only two non-zero summands when we take into account that the product vanishes if any factor vanishes. Indeed, the only non-trivial summands are those with σ satisfying $\sigma_1 = 4$, $\sigma_2 = 3$, and either $\sigma_3 = 1$ in which case $\sigma_4 = 2$, or $\sigma_3 = 2$ in which case $\sigma_4 = 1$. The sign of the first permutation, $(4, 3, 1, 2)$ is -1, as the number of inversions is $2 + 2 + 1$, and that of the second, $(4, 3, 2, 1)$ is 1 as the number of inversions is $3 + 2 + 1$. Thus,

$$\det A = -2 \cdot 3 \cdot 1 \cdot 1 + 2 \cdot 3 \cdot 2 \cdot 2 = 6 \cdot 3 = 18.$$

Since B is upper triangular, the only non-trivial summand corresponds to the identity permutation $(1, 2, 3, 4)$ (namely we need $\sigma_k \leq k$ for all k, but the injectivity of σ means $\sigma_k = k$ in this case), which has no inversions, so has sign 1, and thus $\det B = +1 \cdot 1 \cdot 1 \cdot 1 = 1$. Since $\det(AB) = \det(A)\det(B)$ for all matrices A, B, in this case we have $\det(AB) = 18$ as well.

(b) (3 points) Suppose $A : V \to W$ is linear where V, W are finite dimensional real vector spaces. Let $N(A) = \{x \in V : \ Ax = 0\}$ and $R(A) = \{Ax : \ x \in V\} \subset W$. Show that $\dim N(A) + \dim R(A) = \dim V$.

Note: If you want, you may use matrices, but be specific about the correspondence between matrices and operators. Also, this problem works over any field.

Solution: First, $N(A)$ is a subspace of V, and V is finite dimensional, so there is a basis $e_1, \dots, e_k$ of $N(A)$ (with possibly $k = 0$), and this can be extended to a basis $e_1, \dots, e_k, e_{k+1}, \dots, e_n$ of V. We claim that $Ae_{k+1}, \dots Ae_n$ is a basis of $R(A)$, which will finish the problem since in this case $\dim R(A) = n - k$, $\dim N(A) = k$, $\dim V = n$.

First, $Ae_{k+1}, \dots Ae_n$ span $R(A)$ because any element y of $R(A)$ is of the form $y = Ax = A \sum_{j=1}^n c_j e_j$ (using that $e_1, \dots, e_n$ is a basis of V), thus $y = \sum_{j=1}^n c_j Ae_j = \sum_{j=k+1}^n c_j Ae_j$, where the penultimate step used $e_j \in N(A)$ for $j \leq k$, so $Ae_j = 0$. Thus, $Ae_{k+1}, \dots Ae_n$ span $R(A)$.

On the other hand, $Ae_{k+1}, \dots Ae_n$ are linearly independent for if $\sum_{j=k+1}^n c_j Ae_j = 0$ then $0 = A \sum_{j=k+1}^n c_j e_j$, so $\sum_{j=k+1}^n c_j e_j \in N(A)$, so $\sum_{j=k+1}^n c_j e_j = \sum_{j=1}^k d_j e_j$, so $\sum_{j=1}^n c_j e_j = 0$ if we let $c_j = -d_j$ for $j \leq k$. But $e_1, \dots, e_n$ are linearly independent by construction, so $c_j = 0$ for all j, and thus we conclude that $Ae_{k+1}, \dots Ae_n$ are linearly independent, hence they give a basis for $R(A)$, proving the claim, and thus completing the proof of $\dim N(A) + \dim R(A) = \dim V$.

2 (a) ($2\frac{1}{2}$ points): Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and let a be a given point of $\mathbb{R}^n$. Give the proof that if there is $\rho > 0$ such that the partial derivatives $D_j f(x), j = 1, \ldots, n$ exist for $\|x - a\| < \rho$ and are continuous at a, then f is differentiable at a.

Solution: As in lecture let $h = (h_1, \ldots, h_n)^{\mathrm{T}}$ and define $h_j = (h_1, \ldots, h_j, 0, \ldots, 0)$ for $j = 1, \ldots, n$, and $h_0 = 0$. Then

$$f(a+h) - f(a) = \sum_{j=1}^{n} (f(a+h_j) - f(a+h_{j-1})) = \sum_{j=1}^{n} h_j D_j f(h_{j-1} + \theta_j h_j \underset{\sim}{e}_j)$$

for some $\theta_j \in (0, 1)$ by the mean-value theorem from 1-variable calculus. Thus for $0 < \|h\| < \rho$ we have

$$\begin{aligned}\|f(a+h) - f(a) - \sum_{j=1}^{n} h_j D_j f(a)\| &= \|\sum_{j=1}^{n} \frac{h_j}{\|h\|}(D_j f(a + h_{j-1} + \theta_j h_j \underset{\sim}{e}_j) - D_j f(a))\| \\ &\le \sum_{j=1}^{n} \|(D_j f(a + h_{j-1} + \theta_j h_j \underset{\sim}{e}_j) - D_j f(a))\| \to 0\end{aligned}$$

as $h \to \underline{0}$ because $D_j f(x)$ is continuous at $x = a$.

(b) ($3\frac{1}{2}$ points): State (without proof) the Lagrange multiplier theorem, and use it (together with any other theorems from lecture that you need) to find a point where the function $xy + z^3$ takes its maximum subject to the constraint that $x^4 + y^4 + z^4 = 1$, and justify your answer.

Note: Your discussion should include the reason that the maximum exists.

Solution: The Lagrange multiplier theorem states that if $g_1, \ldots, g_k$ are C^1 functions on $U \subset \mathbb{R}^n$ open, with linearly independent differentials on their joint zero set $S = \{x \in U : g_1(x) = \ldots = g_k(x) = 0\}$ then at any critical point x of $f|_S$ for any C^1 function $f : U \to \mathbb{R}^n$, there exists $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, k$, such that $Df(x) = \sum_{j=1}^{k} \lambda_j Dg_j(x)$.

Let $g(x, y, z) = x^4 + y^4 + z^4 - 1$ defined on $\mathbb{R}^3$, and let $S = \{(x, y, z) : \ g(x, y, z) = 0\}$. If $(x, y, z) \in S$, then $x^4 + y^4 + z^4 = 1$ shows that $x^4, y^4, z^4 \le 1$ since all summands are non-negative. Thus, $|x|, |y|, |z| \le 1$, so S is bounded. On the other hand, the map g is C^∞, so in particular continuous, so $g^{-1}(\{0\}) = S$ is closed since $\{0\} \subset \mathbb{R}$ is closed. Correspondingly S is compact (as it is closed and bounded), so any continuous function, such as $f|_S$, attains its maximum and minimum on S.

We have g is C^∞ and $Dg(x, y, z) = (4x^3, 4y^3, 4z^3)$, so the vanishing of Dg means $x = y = z = 0$, and thus Dg does not vanish on S. Correspondingly, by the implicit function theorem, S is a C^∞ submanifold of $\mathbb{R}^3$. Further, any critical points p of $f|_S$, which includes all local maxima and minima, satisfy that $Df(p) = \lambda Dg(p)$ for some $\lambda \in \mathbb{R}$ by the Lagrange multiplier theorem. Since $Df(x, y, z) = (y, x, 3z^2)$, this means that $y = 4\lambda x^3$, $x = 4\lambda y^3$, $3z^2 = 4\lambda z^3$. If $\lambda = 0$, this gives $(x, y, z) = 0$, but this is not in S, so to find critical points of $f|_S$ we may assume $\lambda \ne 0$. Substituting y from the first equation into the second gives $x = 256\lambda^4 x^9$, so either $x = 0$ or $256\lambda^4 x^8 = 1$, i.e. $\lambda x^2 = \pm\frac{1}{4}$; it also gives $y = (4\lambda x^2)x = \pm x$, so $z^4 = 1 - 2x^4$.

If $x = 0$, we have $y = 0$ as well, thus $z = \pm 1$ on S, in which case by the third equation $3 = 4\lambda(\pm 1)^3$, which has a solution $\lambda \in \mathbb{R}$, thus $(0, 0, \pm 1)$ are critical points of $f|_S$ and

$f(0,0,1) = 1$, $f(0,0,-1) = -1$. Note that for a open set O containing $(0,0,\pm1)$, such as $O = \{(x,y,z) : \ z > 0\}$, on $O \cap S$ we have $z = \pm(1-x^4-y^4)^{1/4}$, and thus $f|_S(x,y,z) = xy \pm (1-x^4-y^4)^{3/4}$, with the last term having vanishing first and second derivatives by the chain rule. Thus, the quadratic approximation of f at $(0,0,\pm1)$ is $xy \pm 1$, and $xy = \frac{1}{4}((x+y)^2-(x-y)^2)$ is a quadratic form with mixed type (taking $y = x$ gives a local minimum, $y = -x$ a local maximum), so these points are not local minima or maxima.

On the other hand, if $\lambda x^2 = \pm\frac{1}{4}$, so $x \neq 0$, so $4\lambda = \pm x^{-2}$, the third equation gives $3z^2 = \pm z^3/x^2$, i.e. $z^2(z \pm 3x^2) = 0$, so $z = 0$ or $z = \pm 3x^2$. Now if $z = 0$ (and $x = \pm y$), so $2x^4 = 1$ by being on S, and thus $x, y = \pm 2^{-1/4}$, so $(2^{-1/4}, 2^{-1/4}, 0)$, $(2^{-1/4}, -2^{-1/4}, 0)$, $(-2^{-1/4}, 2^{-1/4}, 0)$ and $(-2^{-1/4}, -2^{-1/4}, 0)$ are critical points of $f|_S$, and $f(2^{-1/4}, 2^{-1/4}, 0) = 2^{-1/2} < 1$ and $f(-2^{-1/4}, -2^{-1/4}, 0) = 2^{-1/2} < 1$, while at the other two crititcal points the value is the negative of these so > -1, so in view of $f(0,0,\pm1)$, these are not maxima/minima of f.

It remains to consider $z = \pm 3x^2$, which gives $2x^4 + 81x^8 = 1$ by being on S, thus $x^4 = \frac{-2\pm\sqrt{4+4\cdot 81}}{2\cdot 81} = \frac{1}{81}(-1 \pm \sqrt{82})$. Since $x^4 \geq 0$, we must have the $+$ sign in $\pm$, and thus x is given by $\pm$ the 4th root of this, while y is $\pm x$. Since $9 < \sqrt{82} < 10$, we obtain a number $x^4 \in (\frac{8}{81}, \frac{9}{81})$, so $x^4 < 1/2$, and $1 - 2x^4 > 0$, hence there is a corresponding point z with $(x,y,z) \in S$, namely $z = \pm(1-2x^4)^{-1/4}$. Rather than computing the values to find the biggest one, note that when $y = \pm x$, $z = \pm 3x^2$, $f(x,y,z) = \pm x^2 \pm 27x^6$, which is the largest for fixed $|x|$ when all signs are $+$, and the smallest when all signs are $-$. Thus, the maximum of f is attained at the point $(x, x, 3x^2)$ when $x = \pm\frac{1}{3}(-1+\sqrt{82})^{1/4}$ and the minimum at $(x, -x, -3x^2)$ when $x = \pm\frac{1}{3}(-1+\sqrt{82})^{1/4}$.

3(a) (3 points): Suppose (X, d_X) and (Y, d_Y) are metric spaces (if you wish, you may assume $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ with the relative metric). Show that $f : X \to Y$ is continuous if and only if for all $U \subset Y$ open, $f^{-1}(U) = \{x \in X : \ f(x) \in U\}$ is open.

Solution: Suppose first that f is continuous, and let $U \subset Y$ be open. Suppose $x \in f^{-1}(U)$, i.e. $f(x) \in U$. Then as U is open there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset U$, i.e. $d_Y(y, f(x)) < \varepsilon$ implies $y \in U$. But f is continuous, so there is $\delta > 0$ such that $d_X(x', x) < \delta$ implies $d_Y(f(x'), f(x)) < \varepsilon$. Correspondingly, if $x' \in B_\delta(x)$ then $f(x') \in B_\varepsilon(f(x)) \subset U$, and thus $x' \in f^{-1}(U)$. Thus, $B_\delta(x) \subset f^{-1}(U)$. Since $x \in f^{-1}(U)$ was arbitrary, this shows that $f^{-1}(U)$ is open.

Conversely, suppose $f : X \to Y$ has the property that the inverse image under f of any open set is open. If $x \in X$ and $\varepsilon > 0$, then $B_\varepsilon(f(x))$ is open in Y, and thus $f^{-1}(B_\varepsilon(f(x)))$ is open in X. Hence, as $x \in f^{-1}(B_\varepsilon(f(x)))$ (since $f(x) \in B_\varepsilon(f(x))$) there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$, i.e. if $x' \in B_\delta(x)$ then $f(x') \in B_\varepsilon(f(x))$, i.e. if $d_X(x'x) < \delta$ then $d_Y(f(x'), f(x)) < \varepsilon$, which proves the continuity of f.

(b) (3 points) Assume $\sin x, \cos x$ are defined as usual by the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ respectively. Then (i) prove $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$, and (ii) prove the identity $\sin(x + a) = \sin x \cos a + \cos x \sin a$.

Hint for (ii): For fixed a define $f_a(x) = \sin(x+a) - \sin x \cos a - \cos x \sin a$ and start by showing that $\frac{d^n}{dx^n} f_a(x)|_{x=0} = 0$ for all $n = 0, 1, 2, \ldots$.

Solution: (i) Both series have infinite radius of convergence (e.g., if $x \neq 0$ we have $|\frac{x^{2n+3}}{(2n+3)!}/(\frac{x^{2n+1}}{(2n+1)!})| = x^2/((2n+3)(2n+2)) \to 0$ as $n \to \infty$, so we have absolute convergence of $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x by the ratio test, and a similar argument holds for the other series). By the relevant theorem from lecture we thus have that all derivatives of the series exist and can be obtained by termwise differentiation, whence $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.

(ii) By the chain rule and (i), $f_a'(x) = \cos(x + a) - \cos x \cos a + \sin x \sin a$, $f_a''(x) = -\sin(x+a)+\sin x \cos a+\cos x \sin a = -f_a(x)$, $f_a'''(x) = -\cos(x+a)+\cos x \cos a-\sin x \sin a = -f_a'(x)$ ..., and in general by induction $f_a^{(2k)}(x) = (-1)^k(\sin(x+a) - \sin x \cos a - \cos x \sin a)$ and $f_a^{(2k+1)}(x) = (-1)^k(\cos(x + a) - \cos x \cos a + \sin x \sin a)$, so $f_a^{(2k)}(0) = (-1)^k(\sin a - 0 - \sin a) = 0$ and $f_a^{(2k+1)}(0) = (-1)^k(\cos a - \cos a + 0) = 0$, so in fact the Taylor series of f_a is identically 0. Also $|f_a^{(k)}(x)| \leq 3$ for each $k = 0, 1, \ldots$ and each $x \in \mathbb{R}$, so $R^k \max_{|x|\leq R} |f_a^{(k)}(x)|/k! \leq 3R^k/k! \to 0$ as $k \to \infty$, and hence for each $R > 0$ we have a fixed M such that $3R^k \max_{|x|\leq R} |f_a^{(k)}(x)|/k! \leq M$ for each $k = 0, 1, 2, \ldots$ and hence by a Theorem of lecture the Taylor series of f_a converges to f_a at every point of $\mathbb{R}$. Thus, f_a is identically 0, which proves the claimed identity.

4(a) ($3\frac{1}{2}$ points): Find all eigenvalues and corresponding eigenvectors for the matrix

$$\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

Solution: Let A be the matrix above. Eigenvalues of A are roots of the polynomial $\det(A - \lambda I)$, and

$$\det(A-\lambda I) = \det\begin{pmatrix} 5-\lambda & -2 & 1 \\ -2 & 2-\lambda & 2 \\ 1 & 2 & 5-\lambda \end{pmatrix}$$
$$= (5-\lambda)^2(2-\lambda) - 4 - 4 - 8(5-\lambda) - (2-\lambda)$$
$$= 50 - 45\lambda + 12\lambda^2 - \lambda^3 - 50 + 9\lambda = -\lambda^3 + 12\lambda - 36\lambda = -\lambda(\lambda-6)^2,$$

so the eigenvalues of A are 0 and 6. The zero eigenspace is just the nullspace; for nicer computations it is convenient to switch rows 1 and 3 then row reduce:

$$\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 2 & 5 \\ -2 & 2 & 2 \\ 5 & -2 & 1 \end{pmatrix} \begin{matrix} r_2 \mapsto r_2 + 2r_1 \\ r_3 \mapsto r_3 - 5r_1 \end{matrix} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 6 & 12 \\ 0 & -12 & -24 \end{pmatrix} \begin{matrix} r_2 \mapsto r_2/6 \\ r_3 \mapsto r_3 + 2r_2 \end{matrix} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, elements of the nullspace satisfy $x_2 = -2x_3$ and $x_1 = -2x_2 - 5x_3 = -x_3$, so they are of the form $x_3(-1,-2,1)^{\mathrm{T}}$, and thus the 0-eigenspace is $\mathrm{Span}(-1,-2,1)^{\mathrm{T}}$. On the other hand, the 6-eigenspace is $N(A-6I)$, so row-reducing $A-6I$ gives

$$\begin{pmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{matrix} r_2 \mapsto r_2 + 2r_1 \\ r_3 \mapsto r_3 + r_1 \end{matrix} \begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so $N(A-6I)$ consists of vectors with $-x_1 - 2x_2 + x_3 = 0$, i.e. $x_1 = -2x_2 + x_3$, so it is $\mathrm{Span}((-2,1,0)^{\mathrm{T}}, (1,0,1)^{\mathrm{T}})$, with the two given vectors linearly independent in view of their last two components.

(b) ($1\frac{1}{2}$ points): Prove that the quadratic form $Q(h) = 5h_1^2 - 4h_1h_2 + 2h_2^2 + 2h_1h_3 + 4h_2h_3 + 6h_3^2$ is positive definite.

Hint: compare Q with the quadratic form of the matrix in part (a).

Solution: Let Q_1 denote the quadratic form of the matrix A above. Then $Q(h) = Q_1(h) + h_3^2$. Now, as the eigenvalues of A are ≥ 0, $Q_1(h) \geq 0$ for all h, and $Q_1(h) = 0$ if and only if $h \in N(A)$. Thus, $Q(h) \geq 0$ (as $h_3^2 \geq 0$), and $= 0$ if and only if $h \in N(A)$ and $h_3 = 0$. But by the above calculation the only element of $N(A)$ with vanishing 3rd component is 0, so Q is positive definite as claimed.

5(a) (3 points): Suppose that $\{M_n\}_{n=1}^\infty$ is a sequence with $M_n \geq 0$ for all n, and $\sum_{n=1}^\infty M_n$ converges. Show that if $\{x_n\}_{n=1}^\infty$ is a sequence in a complete normed vector space $(V, \|.\|)$ (if you wish, you may take $V = \mathbb{R}^m$ with the standard norm) and $\|x_n\| \leq M_n$ for all n, then $\sum_{n=1}^\infty x_n$ converges in V, i.e. $\lim_{k\to\infty} \sum_{n=1}^k x_n$ exists. (This is the Weierstrass M-test.)

Solution: Since $\|x_n\| \leq M_n$ for all n, $\sum_{n=1}^\infty \|x_n\|$ converges (in $\mathbb{R}$) since it is a series with non-negative terms, and its partial sums are bounded by those of $\sum_{n=1}^\infty M_n$, which are in turn bounded by their limit, $\sum_{n=1}^\infty M_n$.

Now, we claim that the partial sums $s_n = \sum_{k=1}^n x_k$ form a Cauchy sequence in V. If we show this, the completeness of V implies that they converge, i.e. that $\sum_{k=1}^\infty x_k$ converges, completing the proof.

But if $n > m$ (with $n < m$ following by relabelling, and $n = m$ being automatic),

$$\|s_n - s_m\| = \|\sum_{k=m+1}^{n} x_k\| \leq \sum_{k=m+1}^{n} \|x_k\|.$$

On the other hand, if $\sigma_n = \sum_{k=1}^n \|x_k\|$, then $\sigma_n - \sigma_m = \sum_{k=m+1}^n \|x_k\|$ as well. Since $\{\sigma_n\}_{n=1}^\infty$ converges, it is Cauchy, so given $\varepsilon > 0$ there is N such that $n, m \geq N$ implies $|\sigma_n - \sigma_m| < \varepsilon$. Thus, for $n, m \geq N$, $\|s_n - s_m\| \leq |\sigma_n - \sigma_m| < \varepsilon$, proving the claimed Cauchy property, and thus completing the proof.

(b) (3 points): Show that the series $\sum_{n=1}^\infty \frac{1}{n^4}\cos(nx)$ converges uniformly to a C^1 function $f(x)$ on $[0, 2\pi]$ and $f'(x) = -\sum_{n=1}^\infty \frac{1}{n^3}\sin(nx)$.

Solution: First, observe that the terms of the series, $f_n(x) = n^{-4}\sin nx$, are C^1. Next, the term-by-term differentiated series has terms g_n satisfying $|g_n(x)| = |n^{-3}\sin nx| \leq 1/n^3$, so $\sup|g_n| \leq 1/n^3$. Let $M_n = 1/n^3$, so $\sum_{n=1}^\infty M_n$ converges. Thus by the Weierstrass M-test, applied in $C([0, 2\pi])$ taking into account that $C([0, 2\pi])$ is complete, $\sum_{n=1}^\infty g_n$ converges to some continuous function g uniformly (i.e. the partial sums so converge). Moreover, $|f_n(x)| \leq n^{-4}$, and $\sum_{n=1}^\infty n^{-4}$ also converges, so the Weierstrass M-test shows that $\sum_{n=1}^\infty f_n$ converges to some continuous function f uniformly. Then by a homework problem, f is differentiable and $f' = g$; indeed, with $G_m = \sum_{n=1}^m g_n$, $F_m = \sum_{n=1}^m f_n$, we have F_m is C^1 (the sum is finite!) and $F_m(x) = F_m(0) + \int_0^x F_m' = F_m(0) + \int_0^x G_m$; since $\lim F_m(0) = f(0)$, and since $G_m \to g$ uniformly so $\lim \int_0^x G_m = \int_0^x g$, $f(x) = \lim F_m(x) = \lim F_m(0) + \lim \int_0^x G_m = f(0) + \int_0^x g$, and then by the fundamental theorem of calculus f is differentiable with derivative g which is continuous, so f is C^1, with derivative $g = \sum_{n=1}^\infty g_n = -\sum_{n=1}^\infty n^{-3}\sin nx$.

6(a) (3 points): Suppose V is a finite dimensional real vector space, and $e_1, \ldots, e_n$ is a basis for V. Show that the linear maps $f_i : V \to \mathbb{R}$, $i = 1, \ldots, n$, defined by $f_i(\sum_{j=1}^n a_j e_j) = a_i$ give a basis (called the dual basis) of $V^* = \mathcal{L}(V, \mathbb{R})$.

Solution: First, the maps f_i are indeed linear, and they are linearly independent for if $\sum_{i=1}^n c_i f_i = 0$, then for any j, $0 = \sum_{i=1}^n c_i f_i(e_j) = c_j$, so the linear combination is trivial. On the other hand, if $\ell \in \mathcal{L}(V, \mathbb{R})$, then

$$\ell(\sum_{j=1}^n a_j e_j) = \sum_{j=1}^n a_j \ell(e_j) = \sum_{j=1}^n f_j(\sum_{k=1}^n a_k e_k)\ell(e_j) = (\sum_{j=1}^n \ell(e_j) f_j)(\sum_{k=1}^n a_k e_k),$$

so $\ell = \sum_{j=1}^n \ell(e_j) f_j$ is a linear combination of the f_j, and thus the f_j span V. Correspondingly, $f_1, \ldots, f_n$ is a basis of V.

(b) (3 points): Suppose that $A \in \mathcal{L}(V, V)$ is linear, V, etc., as above. Show that trace $A = \sum_{j=1}^n f_j(Ae_j)$ is independent of the choice of the basis of V, and if A is symmetric, then trace A is the sum of the eigenvalues of A, counted with multiplicity, i.e. if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues, then trace $A = \sum_{j=1}^k \lambda_j \dim N(A - \lambda_j I)$.

Note: $f_j(Ae_j)$ is the jj entry of the matrix of A in the basis $e_1, \ldots, e_n$, so the trace is the sum of the diagonal entries of the matrix.

Solution: Consider a different basis $e'_1, \ldots, e'_n$ of V, and write $f'_1, \ldots, f'_n$ for the dual basis. Thus, writing $e_j = \sum_{k=1}^n c_{kj} e'_k$, we have $f'_i(e_j) = f'_i(\sum_{k=1}^n c_{kj} e'_k) = c_{ij} = \sum_{k=1}^n c_{ik} f_k(e_j)$; since this is true for all j, thus for all linear combinations of the e_j, we have $f'_i = \sum_{k=1}^n c_{ik} f_k$. Thus, $f_j(Ae_j) = f_j(A \sum_{k=1}^n c_{kj} e'_k) = \sum_{k=1}^n c_{kj} f_j(Ae'_k)$. Summing over j and interchanging the two sums:

$$\sum_{j=1}^n f_j(Ae_j) = \sum_{k=1}^n \sum_{j=1}^n c_{kj} f_j(Ae'_k) = \sum_{k=1}^n f'_k(Ae'_k).$$

Since the two sides are exactly the expression of the trace in two arbitrary bases, we conclude that the trace is defined independently of the choice of basis. If A is symmetric, there is an orthonormal basis consisting of eigenvectors of A, say $e_1, \ldots, e_n$ with $Ae_j = \mu_j e_j$. Then by definition of the trace, trace $A = \sum_{j=1}^n f_j(\mu_j e_j) = \sum_{j=1}^n \mu_j$, as claimed.

7 (6 points extra credit only): Suppose $a_{mn} \geq 0$ for $m, n \geq 1$ integer. Show that the set $\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\}$ is bounded above if and only if for each m, $\sum_{n=1}^{\infty} a_{mn}$ converges and $\{\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} :\ M \geq 1\}$ is bounded above. Show moreover that in this case

$$\sup\Big\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\Big\} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn},$$

where both sums on the right hand side converge, and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}.$$

Solution: Suppose first that for each m, $\sum_{n=1}^{\infty} a_{mn}$ converges and $\{\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} :\ M \geq 1\}$ is bounded above. Let B be a finite set, and let $M = \max\{m :\ \exists n \text{ s.t. } (m,n) \in B\}$. Then for each m, $\{n :\ (m,n) \in B\}$ is a finite set, and (as shown in lecture, or simply because it is bounded by a partial sum which in turn is bounded by the limit of the series since the terms are non-negative) $\sum_{\{n:\ (m,n)\in B\}} a_{mn} \leq \sum_{n=1}^{\infty} a_{mn}$, and thus summing over m,

$$\sum_{(m,n)\in B} a_{mn} = \sum_{m=1}^{M}\sum_{\{n:\ (m,n)\in B\}} a_{mn} \leq \sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} \leq \sup\{\sum_{m=1}^{\tilde{M}}\sum_{n=1}^{\infty} a_{mn} :\ \tilde{M} \geq 1\},$$

giving the claimed boundedness of the set of finite sums, and more over that the sup of the right hand side (set of finite sum of sums in n) is $\geq$ the sup if the left hand side (set of finite sums).

Conversely, suppose that $\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\}$ is bounded above. Then for each m, the set of partial sums $\sum_{n=1}^{N} a_{mn}$ is bounded above since this is a special case of the finite sums, with $B = \{(m,n) :\ n \leq N\}$, and thus, $\sum_{n=1}^{\infty} a_{mn}$ converges (a series with non-negative terms converges if and only if the set of partial sums is bounded above). Now, for any $\varepsilon > 0$ and any m there is N_m such that $|\sum_{n=1}^{\infty} a_{mn} - \sum_{n=1}^{N_m} a_{mn}| < 2^{-m}\varepsilon$ by the definition of convergence. Thus, taking a finite sum in m:

$$|\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} - \sum_{m=1}^{M}\sum_{n=1}^{N_m} a_{mn}| \leq \sum_{m=1}^{M} |\sum_{n=1}^{\infty} a_{mn} - \sum_{n=1}^{N_m} a_{mn}| \leq \sum_{m=1}^{M} \varepsilon 2^{-m} = \varepsilon 2^{-1}\frac{1-2^{-m}}{1-2^{-1}} \leq \varepsilon.$$

But $B = \{(m,n) :\ m \leq M,\ n \leq N_m\}$ is a finite set, and $\sum_{m=1}^{M}\sum_{n=1}^{N_m} a_{mn} = \sum_{\{(m,n)\in B\}} a_{mn}$, thus we deduce that

$$\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} \leq \sum_{\{(m,n)\in B\}} a_{mn} + \varepsilon \leq \varepsilon + \sup\{\sum_{(m,n)\in \tilde{B}} a_{mn},\ \tilde{B} \subset \mathbb{N}^+ \times \mathbb{N}^+,\ \tilde{B} \text{ finite}\},$$

so the set of $\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn}$ is bounded above, procing the claimed equivalence of boundedness, and moreover that for any $\varepsilon > 0$,

$$\sup\{\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} :\ M \geq 1\} \leq \varepsilon + \sup\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\}.$$

Since $\varepsilon > 0$ is arbitrary, we have $\sup\{\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} : M \geq 1\} \leq \sup\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\}$.

Combining these two results, we have, when either of the equivalent conditions hold,

$$\sup\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\} = \sup\{\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} :\ M \geq 1\}.$$

But as $\sum_{n=1}^{\infty} a_{mn} \geq 0$ for all m, as proved in lecture/homework, $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sup\{\sum_{m=1}^{M}\sum_{n=1}^{\infty} a_{mn} :\ M \geq 1\}$, so

$$\sup\left\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\right\} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}.$$

Finally, notice that the role of m and n is interchangeable (e.g. define $b_{nm} = a_{mn}$; the set of finite sums is unchanged under this switch, while the iterated sums reverse order), we have

$$\sup\left\{\sum_{(m,n)\in B} a_{mn},\ B \subset \mathbb{N}^+ \times \mathbb{N}^+,\ B \text{ finite}\right\} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}.$$

Combining these two statements, we deduce that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn},$$

as claimed.

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