Mathematics Department Stanford University
Math 51H Final Examination, December 9, 2013

3 Hours

Solutions

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Q.6	
Q.7	
Q.8	
T/40	

Name (Print Clearly): ____________________

I understand and accept the provisions of the honor code (Signed) ____________________

1 (a) (2 points): Calculate the determinant of

$$\begin{pmatrix} 11 & 12 & 13 & 426 \\ 2001 & 2002 & 2003 & 421 \\ 2 & 1 & 0 & -419 \\ 101 & 101 & 102 & 2000 \end{pmatrix}$$

No calculators: Clearly state all column/row operations.

Solution:

$$\begin{pmatrix} 11 & 12 & 13 & 426 \\ 2001 & 2002 & 2003 & 421 \\ 2 & 1 & 0 & -419 \\ 101 & 101 & 102 & 2000 \end{pmatrix} \begin{pmatrix} c_2 \mapsto c_2 - c_1 \\ c_3 \mapsto c_3 - c_1 \end{pmatrix} \begin{pmatrix} 11 & 1 & 2 & 426 \\ 2001 & 1 & 2 & 421 \\ 2 & -1 & -2 & -419 \\ 101 & 0 & 1 & 2000 \end{pmatrix}$$

$$\begin{pmatrix} r_2 \mapsto r_2 - r_1 \\ r_3 \mapsto r_3 - r_1 \end{pmatrix} \begin{pmatrix} 11 & 1 & 2 & 426 \\ 1990 & 0 & 0 & -5 \\ 13 & 0 & 0 & 7 \\ 101 & 0 & 1 & 2000 \end{pmatrix}$$

Now none of the above operations changes the determinant so we can just compute the determinant of the last matrix above, and expanding this down the second column gives

$$-\det \begin{pmatrix} 1990 & 0 & -5 \\ 13 & 0 & 7 \\ 101 & 1 & 2000 \end{pmatrix} = +\det \begin{pmatrix} 1990 & -5 \\ 13 & 7 \end{pmatrix} = 7 \times 1990 + 5 \times 13 = 13,930 + 65 = 13,995.$$

(b) (3 points): Find the matrix of the orthogonal projection onto the plane $V = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$.

Hint: Start by finding the orthogonal projection onto the (1-dimensional) normal space $V^\perp$.

The given plane V is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$, i.e. the plane is the set of all points orthogonal to the vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, and so $V^\perp$ is the 1-dimensional space spanned by the unit vector $\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, and the othogonal projection onto the normal space is the map taking the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the vector $\frac{1}{6} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ which is the linear transformation with matrix $\frac{1}{6} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} (2, 1, -1) = \frac{1}{6} \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$, and the orthogonal projection onto V has matrix $I-$ this matrix; i.e. $\frac{1}{6} \begin{pmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{pmatrix}$.

2. (a) (2 points): If $u : \mathbb{R}^n \to \mathbb{R}$ is C^1 and if $\gamma : \mathbb{R} \to \mathbb{R}^n$ is also C^1, prove that the velocity vector $\Gamma'(t)$ of the curve $\Gamma(t) = \begin{pmatrix} \gamma(t) \\ u(\gamma(t)) \end{pmatrix}$ is orthogonal to the vector $\begin{pmatrix} \nabla u(\gamma(t)) \\ -1 \end{pmatrix}$ for each $t \in \mathbb{R}$.

Solution: By the chain rule $\frac{d}{dt}(u(\gamma(t))) = \sum_{j=1}^n D_j u(\gamma(t))\gamma_j'(t) = \gamma'(t) \cdot \nabla u(\gamma(t))$, so $\Gamma'(t) = \begin{pmatrix} \gamma'(t) \\ \gamma'(t) \cdot \nabla u(\gamma(t)) \end{pmatrix}$, and hence $\Gamma'(t) \cdot \begin{pmatrix} \nabla u(\gamma(t)) \\ -1 \end{pmatrix} = \nabla u(\gamma(t)) \cdot \gamma'(t) - \nabla u(\gamma(t)) \cdot \gamma'(t) = 0$.

(b) (3 points) Let e^x be defined as usual by $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ for $x \in \mathbb{R}$. Prove:

(i) $\lim_{x\to 0} |x|^{-p} e^{-1/x^2} = 0$ for each $p > 0$.

Note: You can of course assume, without giving the proof, the standard property $e^{u+v} = e^u e^v$ (so in particular $e^{-u} = 1/e^u$).

(ii) If $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$, find the Taylor series $\sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n$ of f.

Hint for (ii): Start by checking (by induction on n) that for $x \neq 0$ each derivative $f^{(n)}(x)$ has the form $p_n(1/x)e^{-1/x^2}$, where p_n is a polynomial.

Solution (i): Observe first that, for $y > 0$, $e^y = \sum_{n=0}^\infty \frac{y^n}{n!} \geq \frac{y^q}{q!}$ for each $q = 1, 2, \ldots$, so in particular $e^{-1/x^2} \leq q! x^{2q}$ for any $x \neq 0$ and any $q = 1, 2, \ldots$, and hence $|x|^{-p} e^{\frac{-1}{x^2}} \leq q! |x|^{2q-p} \to 0$ as $x \to 0$ if we take $q > p/2$.

Solution (ii): Let P_n be the proposition that the hint is true, $n = 1, 2, \ldots$. By the chain rule $f'(x) = 2x^{-3} e^{-1/x^2}$ for $x \neq 0$, so P_1 is true with $p_1(t) = 2t^3$. If P_n is true then we have $f^{(n)}(x) = p_n(1/x)e^{-1/x^2}$ for $x \neq 0$, and by the product rule for differentiation we get $f^{(n+1)}(x) = (2x^{-3} p_n(1/x) - x^{-2} p_n'(1/x))e^{-1/x^2}$, so P_{n+1} is true with $p_{n+1}(t) = 2t^3 p_n(t) - t^2 p_n'(t)$. Now by (i) all derivatives $f^{(n)}(0) = 0$ because (i) implies $f^{(n+1)}(0) = \lim_{x\to 0} x^{-1}(f^{(n)}(x) - f^{(n)}(0)) = \lim_{x\to 0} x^{-1} p_n(1/x) e^{-1/x^2} = 0$ (and the limit does exist by induction on n starting at $n = 0$). Hence the Taylor series is 0 (the identically zero function).

3 (a) (2 points): Define the term "open set" in $\mathbb{R}^n$, and prove that the intersection $U \cap V$ of 2 open sets U, V is again an open set.

Solution: Let $(x_0, y_0) \in U \cap V$. Then since $(x_0, y_0) \in U$ there is $\delta_1 > 0$ such that the ball $B_{\delta_1}(x_0, y_0) \subset U$ and similarly there is a ball $B_{\delta_2}(x_0, y_0) \subset V$ for some $\delta_2 > 0$, and so taking $\delta = \min\{\delta_1, \delta_2\} (> 0)$ we have $B_\delta(x_0, y_0) \subset$ both U and V; i.e. $B_\delta(x_0, y_0) \subset U \cap V$.

3 (b) (3 points): If $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ are both continuous, and if $S = \{\underline{x} \in \mathbb{R}^n : \varphi(\underline{x}) = 0\}$ is bounded, prove there is a point $\underline{a} \in S$ such that $f(\underline{x}) \leq f(\underline{a})\ \forall\, \underline{x} \in S$.

Solution: We claim that S is closed. Let y be a limit point of S, so there is a sequence $\underline{x}_k \to y$ with $\underline{x}_k \in S$ for each k. Then $\varphi(\underline{x}_k) = 0$ and by continuity of φ we have $\varphi(y) = \lim_{k\to\infty} \varphi(\underline{x}_k) = 0$, so $y \in S$ and we have shown that S is closed. Thus S is a closed bounded set (i.e. a compact set), and hence by a theorem from lecture $f|S$ attains its maximum value somewhere on S; that is, there is a point $\underline{a} \in S$ such that $f(\underline{x}) \leq f(\underline{a})$ for each $\underline{x} \in S$.

4(a) (3 points): State (without proof) the Spectral Theorem for a real symmetric $n \times n$ matrix A, and use it to prove that for a given quadratic form $H(\underset{\sim}{x}) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ ($a_{ij} = a_{ji}$ real) there is a change of coordinates $y = Q^{\mathrm{T}}\underset{\sim}{x}$ with Q orthogonal (i.e. $Q^TQ = QQ^T = I$) such that the quadratic form $H(\underset{\sim}{x})$ is transformed to an expression of the form $\sum_{j=1}^{n} \lambda_j y_j^2$ for suitable real $\lambda_1, \ldots, \lambda_n$.

Solution: The spectral theorem states that if A is a symmetric $n \times n$ matrix then there is an othonormal basis $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ for $\mathbb{R}^n$ such that for each j there is a real λ_j with $A\underset{\sim}{v}_j = \lambda_j\underset{\sim}{v}_j$ (i.e. each $\underset{\sim}{v}_j$ is an eigenvector of A).

Let Q be the matrix with columns $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ and observe that the j'th column of AQ is then $A\underset{\sim}{v}_j = \lambda_j\underset{\sim}{v}_j$ and hence $Q^{\mathrm{T}}(AQ)$ has entry $\underset{\sim}{v}_i \cdot (\lambda_j\underset{\sim}{v}_j)$ in the i'th row and j'th column; i.e. $\lambda_j\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$. That is Q^TAQ is the diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ down the leading diagonal. Observe also that the entry of $Q^{\mathrm{T}}Q$ in the i'th row and j'th column is $\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j = \delta_{ij}$; that is $Q^{\mathrm{T}}Q = I$, so Q is indeed an orthogonal matrix.

The quadratic form $\sum_{i,j} a_{ij}x_ix_j = \underset{\sim}{x}^{\mathrm{T}}A\underset{\sim}{x}$, and with $y = Q^{\mathrm{T}}\underset{\sim}{x}$ (i.e. $\underset{\sim}{x} = Qy$), this is $y^{\mathrm{T}}Q^{\mathrm{T}}AQy = y^{\mathrm{T}}Dy$, where D is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$ down the leading diagonal, so in terms of y the quadratic form is just $\sum_{j=1}^{n} \lambda_j y_j^2$ as claimed.

(b) (2 points). Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \underset{r_3 \mapsto \frac{1}{2}r_3}{} \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \overset{r_1 \mapsto r_1 + r_3}{} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$r_1 \mapsto r_1 - 3r_2 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

so the inverse is $\begin{pmatrix} 1 & -3 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

5(a) (2 points): Give the "(ε, δ) definition" of continuity of a function $f : (a, b) \to \mathbb{R}$ at a point $c \in (a, b)$, and using the definition prove that if $f : (0, 1) \to \mathbb{R}$ is continuous at a point $c \in (0, 1)$ and if $f(c) = 1$ then there is $\delta > 0$ such that $f(x) > \frac{1}{2}$ for all $x \in (c - \delta, c + \delta)$.

Solution: Definition: For each $\varepsilon > 0$ there is a $\delta \in (0, \min\{c, 1 - c\})$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$. Thus $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ whenever $|x - c| < \delta$, so in particular using this with $f(c) = 1$ and $\varepsilon = \frac{1}{2}$ we have that there is a $\delta > 0$ such that $\frac{1}{2} < f(x)$ whenever $|x - c| < \delta$.

5(b) (3 points): Prove that the function $f(x, y) = 1 - 2x - y + 4x^2 + 4xy + 2y^2 + 3xy \sin xy$ has a critical point at $(x, y) = (\frac{1}{4}, 0)$ and that f has a local minimum there.

Solution: The gradient $\nabla f(x, 0)$ is $(-2 + 8x, -1 + 4x)^{\mathrm{T}} = \underset{\sim}{0}$ at $x = \frac{1}{4}$, so $(x, y) = (\frac{1}{4}, 0)$ is a critical point as claimed. Now the Hessian at $(x, y) = (\frac{1}{4}, 0)$ is $\begin{pmatrix} 8 & 4 \\ 4 & 4 + \frac{6}{16} \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 4 & \frac{35}{8} \end{pmatrix}$ and hence the Hessian quadratic form is $8y_1^2 + (35/8)y_2^2 + 8y_1y_2 \geq 4y_1^2 + 4(y_1^2 + y_2^2 + 2y_1y_2) = 4y_1^2 + (y_1 + y_2)^2 > 0$ for $(y_1, y_2) \neq (0, 0)$, so by the second derivative test f has a strict local min at $(x, y) = (\frac{1}{4}, 0)$. (We proved generally that if $\underset{\sim}{a}$ is a critical point f and if the Hessian of f at $\underset{\sim}{a}$ is positive definite, then the function has a strict local minimum at $\underset{\sim}{a}$.)

6 (a) (2 points): Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $\underset{\sim}{v}_1 = (1,1,0,0)^{\mathrm{T}}, \underset{\sim}{v}_2 = (0,1,1,0)^{\mathrm{T}}, v_3 = (0,0,1,1)^{\mathrm{T}}$.

Solution: It is better to use the order $\underset{\sim}{v}_1, \underset{\sim}{v}_3, \underset{\sim}{v}_2$, because $\underset{\sim}{v}_1, \underset{\sim}{v}_3$ are already orthogonal, and so the normalized vectors $\underset{\sim}{w}_1 = \frac{1}{\sqrt{2}} v_1, \underset{\sim}{w}_2 = \frac{1}{\sqrt{2}} \underset{\sim}{v}_3$, are already orthonormal, and the Gram-Schmidt process requires only one further step $\underset{\sim}{w}_3 = \|\underset{\sim}{v}_2 - \underset{\sim}{w}_1 \cdot \underset{\sim}{v}_2 \underset{\sim}{w}_1 - \underset{\sim}{w}_2 \cdot \underset{\sim}{v}_2 \underset{\sim}{w}_2\|^{-1}(\underset{\sim}{v}_2 - \underset{\sim}{w}_1 \cdot \underset{\sim}{v}_2 \underset{\sim}{w}_1 - \underset{\sim}{w}_2 \cdot \underset{\sim}{v}_2 \underset{\sim}{w}_2) = \|(0,1,1,0)^{\mathrm{T}} - (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})^{\mathrm{T}}\|^{-1}((0,1,1,0)^{\mathrm{T}} - (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})^{\mathrm{T}}) = (-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})^{\mathrm{T}}$.
Thus the required orthonormal basis is $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0)^{\mathrm{T}}, (0,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})^{\mathrm{T}}, (-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})^{\mathrm{T}}$.

(b) (3 points): Find the set of all solutions of the inhomogeneous system $A\underset{\sim}{x} = \underset{\sim}{y}$ where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \qquad \underset{\sim}{y} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix}$$

(Give your answer as an affine space.)

Solution: Consider the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 2 & 4 \\ 1 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

To compute the solution set, as in lecture we use elementary row operations on the augmented matrix which reduce A to reduced row echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 & 2 & 4 \\ 1 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \begin{array}{c} r_2 \mapsto r_2 - 2r_1 \\ r_3 \mapsto r_3 - r_1 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) r_3 \mapsto r_3 - r_2 \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -2 & 2 & -2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

$$\begin{array}{r} r_3 \mapsto r_3/2 \\ r_4 \mapsto r_4 - r_3/2 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} r_1 \mapsto r_1 - r_3 \\ r_2 \mapsto r_2 + r_3 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $(x,y,z,u,v)^{\mathrm{T}}$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{y} \iff z = u - v - 1,\ y = -v + 1,\ x = -2u + 2 \iff (x,y,z,u,v)^{\mathrm{T}} = (-2u, -v, u - v, u, v)^{\mathrm{T}} + (2,1,-1,0,0)^T = u(-2,0,1,1,0)^{\mathrm{T}} + v(0,-1,-1,0,1)^{\mathrm{T}} + (2,1,-1,0,0)^T$, where u, v are arbitrary real constants, so the solution set is the 2-dimensional affine space $\mathrm{span}\{(-2,0,1,1,0)^{\mathrm{T}}, (0,-1,-1,0,1)^{\mathrm{T}}\} + (2,1,-1,0,0)^T$.

7(a) (2 points): Find all eigenvalues and corresponding eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Solution: The eigenvalues are the roots of $\det \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$; i.e. $(1-\lambda)^2(2-\lambda) = 0$; i.e. eigenvalues are $\lambda = 1$ (with multiplicity 2) and $\lambda = 2$. If $\lambda = 1$ the eigenvectors are the non-zero solutions of the homogeneous linear system with matrix $\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ which has the null space spanned by $\underset{\sim}{e}_1$; i.e. the set of all eigenvectors is just the set of all non-zero multiples of the vector $\underset{\sim}{e}_1$.

For $\lambda = 2$ the eigenvectors are the non-zero solutions of the homogeneous linear system with matrix $\begin{pmatrix} -1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ which has rref $\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and hence the null space is spanned by $(5,1,1)^{\mathrm{T}}$; i.e. the set of all eigenvectors is just the set of all non-zero multiples of the vector $(5,1,1)^{\mathrm{T}}$.

7 (b) (3 points): Show that the system of two non-linear equations

$$(x^2 - y^2)y + 7x = 1$$
$$(x^2 - y^2)x + 5y = 1$$

has a solution (x, y) with $x^2 + y^2 < 1$.

Hint: Define $f(x,y) = \big(\frac{1}{7}\big(1 - (x^2 - y^2)y\big), \frac{1}{5}\big(1 - (x^2 - y^2)x\big)\big)$ and start by proving that f is a contraction mapping $D \to D$, where $D = \{(x,y) : x^2 + y^2 \le 1\}$.

Solution: With f as in the hint we have $\|f(x,y)\| \le |\frac{1}{7}(1-(x^2-y^2)y)| + |\frac{1}{5}(1-(x^2-y^2)x)| \le \frac{2}{7} + \frac{2}{5} < 1$, so in fact f maps the closed disc D into the open disc $\breve{D}$. Also the derivative matrix $Df(x,y)$ (with columns $D_x f^T(x,y)$ and $D_y f^T(x,y)$) is $\begin{pmatrix} -2xy/7 & (-x^2+3y^2)/7 \\ (-3x^2+y^2)/5 & 2xy/5 \end{pmatrix}$ and so $\|Df(x,y)\|^2 = 4x^2y^2(1/49+1/25) + (3y^2-x^2)^2/49 + (y^2-3x^2)^2/25 \le 4/49+4/25+9/49+9/25 = 13/49+13/25 < 1$ for $x^2+y^2 \le 1$, so since (from lecture) $\|f(x,y) - f(a,b)\| \le \max_{(\xi,\eta)\in D} \|Df(\xi,\eta)\| \|(x,y)-(a,b)\|$ for each $(x,y),(a,b) \in D$, we have shown that f is a contraction. The contraction mapping theorem then tells us that f has a fixed point in D and a fixed point (x,y) of f clearly satisfies the given equations. Notice that the fixed point is actually in the open disk $x^2 + y^2 < 1$ because we proved above that f maps D into the open disk.

8(a) (2 points): Let A be an $n \times n$ real matrix (a_{ij}). Define the adjoint matrix $\operatorname{adj} A$ and give the proof that $A \operatorname{adj} A = (\det A)I$.

Solution: $\operatorname{adj} A$ is the $n \times n$ matrix which has $(-1)^{i+j} \det A_{ji}$ in the i-th row and j-th column, where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A.

From lecture we have the formulae for the expansion of $\det A$ along the j-th row of A:

$$(*) \qquad \textstyle\sum_{k=1}^{n} a_{jk}((-1)^{j+k} \det A_{jk}) = \det A, \quad j = 1, \ldots, n,$$

and hence

$$\textstyle\sum_{k=1}^{n} a_{\ell k}((-1)^{j+k} \det A_{jk}) = 0 \quad \ell \neq j$$

because by $(*)$ it is the expression for determinant of the matrix $\tilde{A}$ which is the same as A except that it has row ℓ of A in both the ℓ-th and the j-th row. Thus

$$\textstyle\sum_{k=1}^{n} a_{ik}((-1)^{j+k} \det A_{jk}) = \det A\, \delta_{ij}, \;\; i, j = 1, \ldots, n.$$

On the other hand the expression on the left of the previous identity is exactly the element which appears in the i-th row and j-th column of $A \operatorname{adj} A$ and the expression on the right is exactly the element which appears in the i-th row and j-th column of $\det A\, I$, so we have proved $A \operatorname{adj} A = \det A\, I$.

8(b) (3 points): Show that $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + z^2 = 1\}$ is a 2-dimensional C^1 manifold and find a point $\underset{\sim}{a} \in S$ at which the function $f(x, y, z) = xyz$ takes its maximum.

Note: You should begin by discussing the existence of such a point $\underset{\sim}{a} \in S$.

Solution: Let $g(x, y, z) = x^2 + 4y^2 + z^2 - 1$, so $S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$, and note that $Dg(x, y, z) = (2x, 8y, 2y) \neq (0, 0, 0)$ on S, hence by a result of lecture (the corollary of the implicit function theorem) S is a 2 dimensional C^1 manifold. S is clearly closed and bounded (indeed $(x, y, z) \in S \Rightarrow x^2 + y^2 + z^2 \leq x^2 + 4y^2 + z^2 \leq 1$ and of course any limit point of S is evidently in S by continuity of g). Thus $f|S$ attains its maximum (since a continuous function on a closed bounded set attains its maximum).

According to the Lagrange multiplier result, at any critical point of $f|S$ (and in particular at any local max/min of $f|S$) we must have $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, where as above $g = x^2 + 4y^2 + z^2 - 1$. Thus at any local max/min of $f|S$ we must have $(yz, xz, xy) = \lambda(2x, 8y, 2z)$; i.e. we have the 3 equations $yz = 2\lambda x, xz = 8\lambda y, xy = 2\lambda z$ and by multiplying the first by x, the second by y, and the third by z we get either $\lambda = 0$ or $x^2 = 4y^2 = z^2$. But $\lambda = 0$ implies that $yz = xz = xy = 0$ which implies that $xyz = 0$ so this cannot happen at a maximum of xyz because there are values where xyz is positive on S and hence the maximum (which exists by the discussion above) must be positive. Thus at a max we have $x^2 = 4y^2 = z^2$, which, since $x^2 + 4y^2 + z^2 = 1$, gives $x^2 = 4y^2 = z^2 = \frac{1}{3}$, and the value of f at any such point is $\pm\frac{1}{6\sqrt{3}}$ so the maximum is $\frac{1}{6\sqrt{3}}$ and is attained at $(x, y, z) = (\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}})$ (and also at several other points, e.g. $(x, y, z) = (\frac{-1}{\sqrt{3}}, \frac{-1}{2\sqrt{3}}, \frac{1}{\sqrt{3}})$).