Math 41 First Exam — October 18, 2012

Name: ________________________________ SUID#: ____________

Circle your section:

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<th>Section</th>
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<tr>
<td>04</td>
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- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.

- Please check that your copy of this exam contains 11 numbered pages and is correctly stapled.

- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.

- You have 2 hours. Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.

- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of a page or one of the extra sheets provided in this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.

- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until Thursday, November 1, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.

- Please sign the following:

  “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

  Signature: ________________________________
1. (12 points) Find each of the following limits, with justification. If the limit does not exist, explain why. If there is an infinite limit, then explain whether it is $\infty$ or $-\infty$.

(a) $\lim_{x \to 1} \left( \frac{2}{x^2 - 1} - \frac{1}{x - 1} \right)$

(b) $\lim_{x \to -2^+} \frac{x \ln(2 + x)}{x^2 + 2x + 2}$
(c) \[ \lim_{x \to 0} (\sin^2 x) \frac{\cos(1/x)}{2} \]
2. (10 points) Mark each statement below as true or false by circling either TRUE or FALSE. *No justification is necessary.*

(a) For any positive $\epsilon$ and any $x$, we have
\[ |(x^2 - 2x + 5) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 1| < \sqrt{\epsilon}. \]

TRUE   FALSE

(b) For any $x$, we have
\[ |x^2 - 1| < 1 \quad \text{whenever} \quad 0 < \left| x - \frac{1}{2} \right| < \frac{1}{4}. \]

TRUE   FALSE

(c) For any positive $\epsilon$, there is a corresponding positive $\delta$ such that
\[ |x^2 - 1| < \epsilon \quad \text{whenever} \quad 0 < \left| x - \frac{1}{2} \right| < \delta. \]

TRUE   FALSE

(d) For any function $f$ and any $a$, if
\[ \lim_{x \to a^+} f(x) \text{ and } \lim_{x \to a^-} f(x) \]
both exist and are equal, then $f$ is continuous at $x = a$.

TRUE   FALSE

(e) For any *odd* function $f$, if
\[ f(0) = \lim_{x \to 0^+} f(x), \]
then $f$ is continuous at $x = 0$.

TRUE   FALSE
3. (12 points) Let \( f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \)

(a) Find the equations of all \textit{vertical} asymptotes of \( f \), or explain why none exist. As justification for each asymptote \( x = a \), calculate both the one-sided limits \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \), showing your reasoning.

(b) Find the equations of all \textit{horizontal} asymptotes of \( f \), or explain why none exist. Justify using limit computations.
4. (8 points) Prove, using precise statements, that there is a real number $x$ between $-1$ and $1$ which is a solution to the equation

$$\sin\left(\frac{\pi x}{2}\right) = \ln(x^2).$$
5. (8 points) Let \( f(x) = \frac{1}{2 + \sqrt{x + 1}} \). Find a formula for \( f'(x) \) using the limit definition of the derivative. Show the steps of your computation.
6. (12 points) The reproduction pattern of a certain species of fruit fly, grown in bottles in a laboratory, depends on the number \( p \) of female flies in the bottle. A researcher determines values of \( S(p) \), the number of daily offspring per female; the chart below shows a few values of \( S(p) \):

<table>
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<tr>
<th>( p )</th>
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<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
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<tbody>
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<td>( S(p) )</td>
<td>5.3</td>
<td>4.9</td>
<td>4.5</td>
<td>4.2</td>
<td>4.0</td>
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(a) Give your best estimate for the value of \( S'(14) \), showing your reasoning, and make sure to specify the units of this quantity.

(b) What is the practical meaning of the quantity \( S'(14) \)? Give a brief but complete one- to two-sentence explanation that is understandable to someone who is not familiar with calculus.
(Problem 6 continued) For easy reference, here again is the setup of the problem: The reproduction pattern of a certain species of fruit fly, grown in bottles in a laboratory, depends on the number \( p \) of female flies in the bottle. A researcher determines values of \( S(p) \), the number of daily offspring per female; the chart below shows a few values of \( S(p) \):

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(c) For this and part (d), let \( g(p) = p \cdot S(p) \). Compute a formula for \( g'(p) \). (Express your answer in terms of quantities such as \( p \), \( S(p) \), and \( S'(p) \).)

(d) State a practical implication of the statement “\( g'(14) \) is positive” using only terminology of the laboratory setting. (You don’t have to estimate the value of \( g'(14) \); just assume it is positive for the purposes of answering this part.)
7. (8 points) Find the derivative, using any method you like. You do not need to simplify your answers.

(a) \( h(x) = \frac{x^3 + \pi x^{1/4} - (\mathrm{e}/x)}{\sqrt{x}} \)

(b) \( f(x) = \frac{\sin x - 2 \cos x}{e^x} \)
8. (7 points) The figure below shows the graph of a function $f$ that has continuous first and second derivatives. The dashed lines are tangent to the graph of $y = f(x)$ at $(1, 1)$ and $(5, 1)$.

List the following quantities in increasing order (from smallest to largest). No justification is necessary.

$f(1) \quad f(3) - f(2) \quad \frac{1}{3} (f(5) - f(2)) \quad$ The number $-2 \quad f'(\frac{1}{2}) \quad f'(1) \quad f'(5)$
9. (12 points) The following is a graph of the function $g$:

(a) Consider the function $g'$, the derivative of $g$. Based on estimating $g'$ from the graph of $g$ above, on what intervals is $g'$ increasing? decreasing? (No justification is necessary.)

(b) Based on the picture of $g$ above, which of the following expressions is a plausible formula for $g(x)$? Circle your answer; no justification is necessary. (You may take it as a given that exactly one of these formulas is the best answer.)

- $\frac{x - 1}{x^2(x - 2)}$
- $\frac{x - 1}{x^{1/3}(x - 2)^2}$
- $\frac{(x - 1)^3}{x^2(x - 2)^3}$
- $\frac{(x - 1)^2}{x^{1/3}(x - 2)^2}$

(c) On the set of axes below, sketch a plausible graph of a function $f$ satisfying all of the following:

- $f$ is continuous on $(-\infty, 2) \cup (2, \infty)$,
- $f'(x) = g(x)$ for all $x$ in the domain of $g$,
- $f$ has vertical asymptote $x = 2$, and
- $f(0) = 0$. 
