1. (20 points) Differentiate, using the method of your choice.

(a) \( f(x) = (\sqrt{x})\cos(x) \)

\[
f'(x) = (\sqrt{x})\cos(x) \left( \frac{1}{2x} \cos(x) - \frac{\sin(x) \ln x}{2} \right)
\]

(b) \( g(x) = \frac{e^{x^2}}{\sec(8x^3)} \)

\[
g'(x) = (e^{x^2} \cos(8x^3))' = 2xe^{x^2} \cos(8x^3) - 24x^2e^{x^2} \sin(8x^3)
\]
(c) \[ Z(x) = \arctan(x - \sqrt{1 + x^2}) \]

\[ Z'(x) = \frac{1 - \frac{x}{\sqrt{1+x^2}}}{1 + (x-\sqrt{1+x^2})^2} = \frac{1}{2x^2+2} \]

(d) \[ h(t) = \arccos(t \cos(t)) \]

\[ h'(t) = \frac{t \cos t}{\sqrt{1-t^2 \cos^2 t}} (- \sin t \ln t + \cos t/t) \]
2. (15 points) (a) Use linear approximation to estimate \( \frac{e^{1.1}}{\sqrt{1.1}} \), showing your reasoning.

Let \( f(x) = \frac{e^x}{\sqrt{x}} \). Then \( f'(x) = e^x(x^{-1/2} - \frac{1}{2}e^{-3/2}) \). The linearization of \( f(x) \) at \( x = 1 \) is

\[
L(x) = f'(1)(x-1) + f(1) = e/2(x-1) + e.
\]

So \( f(1.1) \approx L(1.1) = \frac{21}{20}e \).

(b) Is your estimate in part (a) larger or smaller than the actual value? Justify your answer.

Since \( f''(x) = e^x x^{-5/2}(x^2 - x + \frac{3}{4}) \) is positive when \( x > 1 \), then the graph of \( f(x) \) is concave upwards when \( x > 1 \). It implies that the estimate in (a) will be an underestimate, i.e. smaller than the actual value.
3. (15 points) Air is being pumped into a spherical balloon at a rate of 5 cm$^3$/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm. [The formula for the volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$.]

Let $r = r(t)$ be the radius of the balloon at time $t$ and let $V(t)$ be the corresponding volume. As $V = \frac{4}{3}\pi r^3$, by taking the differentials at both sides, we get

$$\frac{dV}{dt} = \frac{d(4/3\pi r^3)}{dt} = 4\pi r^2 \frac{dr}{dt}.$$  

When $\frac{dV}{dt} = 5$ and $r = \frac{20}{2} = 10$, $\frac{dr}{dt} = \frac{5}{400} = 1/80$ cm/min.
4. (15 points) Consider the curve $x - 3y = 4xy^2$.

(a) Find an equation for the tangent line to the curve at the point $(1, -1)$.

4(a) Differentiating the curve $x - 3y = 4xy^2$ implicitly, we have

$$1 - 3 \frac{dy}{dx} = 4y^2 + 8xy \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1 - 4y^2}{8xy + 3},$$

where we have used the product rule to get the first equality. Evaluating this expression at the point $(x, y) = (1, -1)$ gives us the slope of the tangent line at that point, namely

$$\frac{dy}{dx} \bigg|_{(1, -1)} = \frac{1 - 4(-1)^2}{8(1)(-1) + 3} = \frac{-3}{-5} = \frac{3}{5}.$$

The point-slope form of the line’s equation is thus given by

$$y + 1 = \frac{3}{5}(x - 1).$$

(b) Show that this curve does not have any horizontal tangent line.

The curve has a horizontal tangent line only at the points where $dy/dx$ vanishes. From the expression we derived in part (a), this is only at the points where

$$\frac{1 - 4y^2}{8xy + 3} = 0 \implies 1 - 4y^2 = 0 \implies y = \pm \frac{1}{2}.$$

However, no points on the curve have $y$-coordinate $\pm 1/2$, for this would imply

$$3y = x - 4xy^2 = x(1 - 4y^2) = 0 \implies y = 0,$$

a contradiction since we assumed $y = \pm 1/2$. This contradiction proves that there are no points on the curve with horizontal tangent lines.
5. (20 points) Let \( f(x) = x\sqrt{4 - x^2} \). For your convenience, the derivative of \( f(x) \) is \( f'(x) = \frac{2(2 - x^2)}{\sqrt{4 - x^2}} \).

(a) Find the domain of \( f(x) \).

The domain is \( \{4 - x^2 \geq 0\} = [-2, 2] \).

(b) Determine the intervals on which \( f(x) \) is increasing and decreasing.

\( f(x) \) is increasing if \( -\sqrt{2} \leq x \leq \sqrt{2} \).

\( f(x) \) is decreasing if \( -2 < x \leq -\sqrt{2} \) or \( \sqrt{2} \leq x < 2 \).
(c) Determine the intervals of concavity for $f(x)$, i.e., the intervals where the graph of $f(x)$ is concave up and the intervals where the graph of $f(x)$ is concave down.

\[ f''(x) = \frac{2x(x^2 - 6)}{(4-x^2)^{3/2}}. \]

The graph of $f(x)$ is concave up when $f''(x) > 0$, which gives $-2 < x < 0$.

The graph of $f(x)$ is concave down when $f''(x) < 0$, which means $0 < x < 2$.

(d) Determine local maxima and local minima of $f(x)$.

The critical points of $f(x)$ are $x = \pm \sqrt{2}$. By second derivative test, we get the local minimal point of $f(x)$ is $x = -\sqrt{2}$, while the local maximal point of $f(x)$ is $x = \sqrt{2}$. 
(e) Using parts (a)-(d), sketch an accurate graph of $f(x)$ on the axes provided.
6. (15 points) Determine the points on the parabola \( y = x^2 + 1 \) that are closest to \((0, 2)\). Justify your reasoning.

Here is a picture:

![Graph of the parabola and the distance to the point (0, 2)](image)

The distance between \((0, 2)\) and a point \((x, y)\) on \( y = x^2 + 1 \) is given by

\[
\sqrt{(x - 0)^2 + (y - 2)^2}.
\]

To find the closest distance, we only need to minimize the square of the distance function \( x^2 + (y-2)^2 \).

Note that \( x^2 = y - 1 \), then we obtain a function of \( y \):

\[
d(y) = (y - 1) + (y - 2)^2 = y^2 - 3y + 3,
\]

with \( y \geq 1 \).

As \( d'(y) = 2y - 3 \), the function only has one critical point \( y = \frac{3}{2} \). By Local-to Global principale, this is a global minimum point since \( d'(y) \) changes from negative to positive at \( y = \frac{3}{2} \). The corresponding points on \( y = x^2 + 1 \) will be \((\sqrt{1/2}, 3/2)\) and \((-\sqrt{1/2}, 3/2)\).