MAXIMAL SUBTREES OF GRAPHS

This document is meant to give a correct proof of the proposition given in class on January 19th. To use Zorn's lemma one needs to show that any chain is bounded. And a chain does not necessarily have a countable index set, a chain is just a totally ordered set. So induction is out of the question (sorry). While it is true that the union of those trees is contractible, the only other proof that I can think of uses things we have not seen in class.

So I am giving you an alternative proposition based on the proof of proposition 1.A.1 of Hatcher's book. I am rewriting it in a way that I think it is easier to understand, and filling in the point-set topology details not proven there. Note that nothing is said about maximality of the subtree, but all that is necessary of that proposition for what follows in class is that $X^0 \subset T$.

Proposition 1. *Every path connected graph X contains a subtree T with $X^0 \subset T$.*

Proof. Let $v \in X^0$ and let $Z_0 = v$.

Let Z_1 be the union of Z_0 and all 1-cells in X that have v as one of the endpoints. Note that when I say 1-cell, I mean including the endpoints.

Inductively, we construct a sequence of subgraphs $Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_i \subset Z_{i+1} \subset \cdots$, where we let Z_{i+1} be the union of Z_i and all 1-cells in $X - Z_i$ that have at least one of the endpoints in Z_i.

Let $Z = \bigcup_{i=0}^{\infty} Z_i$. It is clear that Z is a subcomplex of X. The surprising fact is that $Z = X$, and this is the technical part. Let us prove this:

If $x \in Z$, that means $x \in Z_j$ for some j. But then Z_{j+1} contains an open neighborhood of x, so Z is open in X.

But Z is also closed in X. Assume $x \in \bar{Z} \cap X^0$, that is, x is a vertex. Since X is path connected, x is the endpoint of at least 1-cell in X. Let V be the union of the interior of the 1-cells that have x as an endpoint and $\{x\}$. Then $V \cap Z \neq \emptyset$, from where $x \in Z$ or Z intersects the interior of one of these 1-cells. But in the latter case, since Z is a subcomplex of X, it must contain the whole 1-cell e, and therefore $x \in Z$.

Now assume $x \in \bar{Z}$ is not a vertex. Then x must be in the interior U of some 1-cell e of X. But U is an open set that contains x, so $U \cap Z \neq \emptyset$. Since Z is a subcomplex of X, it must contain the whole 1-cell, thus $x \in Z$.

Since X is path connected, it is connected. Z is open, closed and nonempty, so $Z = X$. Moreover, X has the weak topology with respect to the Z_i's, that is, A is closed in X if and only if $A \cap Z_i$ is closed in Z_i for all i. Let us show this:

If A is closed in X, $A \cap Z_i$ is closed in Z_i for all i because the the Z_i's have the subspace topology from X. Conversely, assume $A \cap Z_i$ is closed in Z_i for all i and let $x \in \bar{A}$. There must be some j such that $x \in Z_j - Z_{j-1}$. By the way the Z_i's are constructed, we must have that x is in the closure of $A \cap Z_{j+1}$ since a sequence converging to x can only get close by being in Z_{j-1}, Z_j or Z_{j+1}. But this is just $A \cap Z_{j+1}$. In particular, $x \in A$, hence A is closed.

Note that this also implies that U is open in X if and only if $U \cap Z_i$ is open in Z_i for all i.

Now, construct the subtree in the following way. Let $T_0 = v$. Then T_1 is the union of T_0 and, for each vertex of $Z_1 - Z_0$, a 1-cell connecting that vertex to v. Inductively, construct T_{i+1} as the union of T_i and, for each vertex of $Z_{i+1} - Z_i$, a 1-cell connecting that vertex to a vertex in T_i.

Now, let $T = \bigcup_{i=0}^{\infty} T_i$. It is clear that T_{i+1} is homotopy equivalent to T_i via the inclusion $j_i : T_i \to T_{i+1}$ and the maps $r_i : T_{i+1} \to T_i$ that retract the new 1-cells not in T_i to the endpoint in T_i. Let $H^{i+1} : T_{i+1} \times I \to T_{i+1}$ be a homotopy from the identity to $j_i r_i$.

Now consider the homotopy $H : T \times I \to T$ defined in the following way. If $x \in T_i$, and $0 \leq t \leq 1/2^i$, then $H(x,t) = x$. If $1/2^i \leq t \leq 1/2^{i-1}$, then $H(x,t) = H^i(x, a(t))$, where $a(t)$ is a linear function that takes $1/2^i$ to 0 and $1/2^{i-1}$ to 1. If $1/2^{i-1} \leq t \leq 1/2^{i-2}$, then $H(x,t) = H^{i-1}(H^i(x,1), b(t))$, where $b(t)$ is a linear function that takes $1/2^{i-1}$ to 0 and $1/2^{i-2}$ to 1. And so on.

Essentially, this homotopy takes an element of T_i and leaves it fixed for a while, then contracts it into T_{i-1} at $t = 1/2^{i-1}$, then into T_{i-2} at $t = 1/2^{i-2}$, etc. until at the end the homotopy takes any point to v when $t = 1$. Let us see why this is continuous:

Since T has the subspace topology from X and X has the weak topology with respect to the Z_i's, T has the weak topology with respect to the T_i's and it is easy to check that $T \times I$ has the weak topology with respect to the $T_i \times I$'s. It is clear that H is continuous restricted to each $T_i \times I$, so given an open set U in T, $H^{-1}(U) \cap (T_i \times I)$ is open in $T_i \times I$ for all i. And therefore $H^{-1}(U)$ is open in $T \times I$. $\square$