MATH 215B HOMEWORK 5 SOLUTIONS

1. (10 marks) Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

Solution

$S^1 \vee S^1$ is a subcomplex of the CW-complex $S^1 \times S^1$. In particular, $(S^1 \times S^1, S^1 \vee S^1)$ is a good pair with $S^1 \times S^1 / S^1 \vee S^1 = S^2$. This last fact is obvious considering the representation of $S^1 \times S^1$ as a square with sides identified, which makes $S^1 \times S^1$ into a CW-complex with one 0-cell, two 1-cells and one 2-cell. Let $q : S^1 \times S^1 \to S^2$ be the quotient map. We look at the following piece of the long exact sequence for this pair:

$$\ldots \to H_2(S^1 \vee S^1) \to H_2(S^1 \times S^1) \xrightarrow{q_*} H_2(S^2) \to \ldots$$

$S^1 \vee S^1$ is a 1-dimensional CW-complex, so $H_2(S^1 \vee S^1) = 0$ and therefore q_* is injective. By example 2.36, $H_2(S^1 \times S^1) \cong \mathbb{Z}$. Recall $H_2(S^2) \cong \mathbb{Z}$ so q_* defines an injective map $\mathbb{Z} \to \mathbb{Z}$. Thus q_* can not be the zero map and so q is not nullhomotopic.

This is enough to show q is not nullhomotopic, but the exercise says that q_* induces an isomorphism on H_2. This is true because the next terms in the long sequence are $H_2(S^2) \to H_1(S^1 \vee S^1) \to H_1(S^1 \times S^1)$, where the second map is induced by the inclusion. The inclusion is a cellular map, so we will look at the map $H_1^{CW}(S^1 \vee S^1) \to H_1^{CW}(S^1 \times S^1)$. This map takes the two 1-cells of $S^1 \vee S^1$ to the two 1-cells of $S^1 \times S^1$, so it is an isomorphism. Since maps induced by cellular maps are natural with respect to the isomorphism of cellular homology and singular homology, the map $H_1(S^1 \vee S^1) \to H_1(S^1 \times S^1)$ is an isomorphism. This implies the map $H_2(S^2) \to H_1(S^1 \vee S^1)$ is the zero map (since the image must be the kernel of the next), and therefore the image of q_* is the kernel of the zero map, that is, the whole $H_2(S^2)$.

Let $p : \mathbb{R}^2 \to S^1 \times S^1$ be the universal cover of $S^1 \times S^1$. Given a map $f : S^2 \to S^1 \times S^1$, we have $f_*(\pi_1(S^2, x_0)) = p_*(\pi_1(\mathbb{R}^2, x_1))$ for any $x_0 \in S^2$ and $x_1 \in \mathbb{R}^2$ with $p(x_1) = f(x_0)$, since S^2 is simply-connected and $\mathbb{R}^2$ is contractible. Therefore, f has a lift $g : S^2 \to \mathbb{R}^2$ which is nullhomotopic because $\mathbb{R}^2$ is contractible, hence f is nullhomotopic.

2. (8 marks) Let $f : S^n \to S^n$ be a continuous map such that $||f(p) - p|| < 1$ for all $p \in S^n$, where $||\,||$ denotes the canonical norm in $\mathbb{R}^{n+1}$. Show that f is surjective.

Solution

f is homotopic to the identity by the following homotopy:

$$h_t(p) = \frac{tp + (1-t)f(p)}{||tp + (1-t)f(p)||}$$

Clearly, $h_0 = f$ and $h_1 = \mathbb{1}_{S^n}$. To check that it is well-defined, note that

$$\begin{aligned} ||tp + (1-t)f(p)|| &= ||f(p) + t(p - f(p))|| \\ &\geq ||f(p)|| - ||t(p - f(p))|| \\ &= 1 - t||p - f(p)|| \\ &\geq 1 - ||p - f(p)|| \\ &> 1 - 1 && \text{by our hypothesis about } f \\ &= 0 \end{aligned}$$

f is homotopic to $\mathbb{1}$, so $\deg f = 1$.

If f were not surjective, then we could choose a point $x \in S^n - \mathrm{im}(f)$; and composing f with a deformation-retraction of $S^n - x$ obtain a nullhomotopy of f. So in this case f would have degree 0. But the degree of f is 1, so f must be surjective.

3. A continuous map $f : X \to Y$ between CW complexes is called cellular if $f(X^n) \subset Y^n$ for all $n \geq 0$, where X^n, Y^n denote the n-skeleton of X and Y, respectively.

(a) (4 marks) Show that f induces a map between the cellular chain complexes of X and Y. Deduce from this fact that f induces maps in cellular homology, $f_*^{CW} : H_k^{CW}(X) \to H_k^{CW}(Y)$ for all $k \geq 0$.
(b) (8 marks) Show that this map is natural with respect to the isomorphism of singular and cellular homology, that is, if $\Phi_X : H_k(X) \to H_k^{CW}(X)$ is the isomorphism given in the proof of theorem 2.35 in page 139 of Hatcher's book (or in class), then $f_*^{CW}\Phi_X = \Phi_Y f_*$.
(c) (6 marks) Using parts (a) and (b), and a suitable CW-structure on S^n, show that a reflection $r : S^n \to S^n$ across a hyperplane in $\mathbb{R}^{n+1}$ that contains the origin has degree -1.

Solution

(a) Recall that the cellular chain groups are defined as $C_k^{CW}(X) := H_k(X^k, X^{k-1})$. f induces a homomorphism $f_* : H_k(X^k, X^{k-1}) \to H_k(Y^k, Y^{k-1})$, and we will call it $f_\#^{CW}$. This homomorphism commutes with the cellular boundary maps:

$$\begin{tikzcd}
C_k^{CW}(X) := \arrow[d, dashed, "f_\#^{CW}"] & H_k(X^k, X^{k-1}) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{k-1}(X^{k-1}) \arrow[r, "j"] \arrow[d, "f_*"] & H_{k-1}(X^{k-1}, X^{k-2}) \arrow[d, "f_*"] & =: C_{k-1}^{CW}(X) \arrow[d, dashed, "f_\#^{CW}"] \\
C_k^{CW}(Y) := & H_k(Y^k, Y^{k-1}) \arrow[r, "\partial"] & H_{k-1}(Y^{k-1}) \arrow[r, "j"] & H_{k-1}(Y^{k-1}, Y^{k-2}) & =: C_{k-1}^{CW}(Y)
\end{tikzcd}$$

The top and bottom rows are the cellular boundary maps. The horizontal arrows are part of long exact sequences of pairs, and the two middle squares commute because the long exact sequence of a pair is natural.

So we have induced homomorphisms $f_\#^{CW}$ of cellular chain complexes. Just as in simplicial or singular homology, this induces homomorphisms f_*^{CW} on cellular homology.

(b) Recall the diagram from page 139 of Hatcher:

$$\begin{tikzcd}
 & & & 0 \\
0 \arrow[dr, "i"] & & H_n(X^{n+1}) \arrow[ur, "j"] \arrow[r, "i", "\approx"'] & H_n(X) \\
 & H_n(X^n) \arrow[ur, "i"] \arrow[dr, "j"] & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) &
\end{tikzcd}$$

The maps labelled i and j come from inclusions of the form $X^n \hookrightarrow X^{n+1}$ and $(X^n, \emptyset) \hookrightarrow (X^n, X^{n-1})$, and the diagonals are short exact sequences of pairs. The isomorphism between singular homology and cellular homology is defined as follows: Starting with a class $x \in H_n(X)$, there is a unique preimage (under i) in $H_n(X^{n+1})$. Choose one preimage of that in $H_n(X^n)$ and call it y. Then $j(y) \in H_n(X^n, X^{n-1})$ is a cellular chain. Theorem 2.35 says that $j(y)$ is in fact a cycle, so it determines a cellular homology class $[j(y)]$, and this class is unique despite the freedom we had in choosing y. The homomorphism $x \mapsto [j(y)]$ is the isomorphism between singular and cellular homology.

Now since f is cellular, it restricts to maps of pairs $f : (X^n, X^{n-1}) \to (Y^n, Y^{n-1})$ for all n, and all the maps in the above diagram are natural with respect to the induced maps f_*. In particular, we have a commutative square:

$$\begin{tikzcd}
H_n(X^n) \arrow[r, "i"] \arrow[d, "f_*"] & H_n(X) \arrow[d, "f_*"] \\
H_n(Y^n) \arrow[r, "i"] & H_n(Y)
\end{tikzcd}$$

If $y \in H_n(X^n)$ is a preimage of $x \in H_n(X)$, then $f_*(y) \in H_n(Y^n)$ is a preimage of $f_*(x) \in H_n(Y)$. And by naturality of j, we have that $[j(f_*(y))] = [f_*(j(y))] = f_*([j(y)])$. So f_* commutes with the isomorphism between singular and cellular homology.

(c) Given the reflection r, the points fixed by r is an equatorial $(n-1)$-sphere. For simplicity's sake we will give it a structure with one 0-cell and one $(n-1)$-cell, attached by the unique map $S^{n-1} \to *$. The remainder of S^n is two hemispheres which are n-cells, whose attaching maps are the identity $S^{n-1} \to S^{n-1}$. Our cellular chain complex looks like this:

$$\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$$

Choosing generators for the cellular chain groups is the same as choosing orientations on the cells. Let us choose orientations for the n-cells so that r takes the orientation on one n-cell to the orientation on the other. And the attaching maps of the n-cells are identical. We now know enough to write down integral matrices for the homomorphisms in the following diagram, which displays the chain complex homomorphism $r_\#$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\,1\;1\,)"] \arrow[d, "r_\#", "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"'] & \mathbb{Z} \arrow[r] \arrow[d, "r_\#", "\mathbb{1}"'] & \cdots \\
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\,1\;1\,)"] & \mathbb{Z} \arrow[r] & \cdots
\end{tikzcd}$$

The cycle $(1,-1) \in C_n^{CW}(S^n)$ represents a generator of the homology, and $r_\#$ takes it to $(-1,1)$. Therefore $r_* : H_n^{CW}(S^n) \to H_n^{CW}(S^n)$ is multiplication by -1. So r is of degree -1.

4. (12 marks) Show that $\tilde{H}_i(S^n - X) \cong \tilde{H}_{n-i-1}(X)$ when X is homeomorphic to a finite connected graph. [First do the case that the graph is a tree.]

Solution

First we prove it when X is homeomorphic to a tree T via a map $h : T \to X$. Since T is contractible, we need to show that $\tilde{H}_i(S^n - X) = 0$ for all i.

We will use induction on the number k of vertices of T. If $k = 0$, then $X = D^0$ and by proposition 2.B.1, $\tilde{H}_i(S^n - X) = 0$ for all i. If $k = 1$, then X is homeomorphic to D^1 and by proposition 2.B.1., $\tilde{H}_i(S^n - X) = 0$ for all i. Now assume the result is true for a tree with k vertices. If T is a tree with $k+1$ vertices, pick a vertex v_0 that has only one edge f coming out of it. This makes $T = T' \cup f$ where T' is a tree with k vertices, and $T' \cap f = v_0$. Let $Y = h(T)$ and $e = h(f)$.

We use Mayer-Vietoris with $A = S^n - e$ and $B = S^n - Y$. Then $A \cap B = S^n - h(v_0)$ and $A \cup B = S^n - X$.

$$\ldots \to \tilde{H}_{i+1}(S^n - h(v_0)) \to \tilde{H}_i(S^n - X) \to \tilde{H}_i(S^n - e) \oplus \tilde{H}_i(S^n - Y) \to \ldots$$

By the case $k = 0$, $\tilde{H}_{i+1}(S^n - h(v_0)) = 0$ and by the inductive assumption and $k = 1$, $\tilde{H}_i(S^n - e) \oplus \tilde{H}_i(S^n - Y) = 0$. Therefore we have proved that $\tilde{H}_i(S^n - X) = 0$ for any i and for any X homeomorphic to a finite tree, as we wanted.

Now assume that X is homeomorphic to a finite graph G via a map $h : G \to X$. Let T be a maximal subtree of G. We will show the result by induction on the number m of edges of G that are not on T. If $m = 0$, then G is a tree and so the result holds by our previous work.

Assume it holds for m, and let G be a graph with $m+1$ edges not in the maximal subtree. Let f be one of those edges, so we can write $G = G' \cup f$, and G' is a graph with m edges not in the maximal subtree and $G' \cap f$ is two vertices v_0 and v_1 (it must be two since the maximal tree contains all the vertices). Let $Y = h(G')$, $e = h(f)$ and $R = h(v_0 \amalg v_1)$.

By the induction hypothesis, $\tilde{H}_i(S^n - Y) \cong \tilde{H}_{n-i-1}(Y) = \begin{cases} \mathbb{Z}^m & \text{if } n - i - 1 = 1 \\ 0 & \text{otherwise} \end{cases}$ because Y is a 1-dimensional complex and so it only has nonzero reduced homology in dimension 1. In dimension 1, it is the abelianization of the fundamental group, which is a free group of rank m, so it is the free abelian group of rank m. And so $S^n - Y$ only has nonzero reduced homology in degree $i = n - 2$.

Now we use Mayer-Vietoris with $A = S^n - e$ and $B = S^n - Y$, so that $A \cap B = S^n - R$ and $A \cup B = S^n - X$.

$$\ldots \to \tilde{H}_{i+1}(S^n - e) \oplus \tilde{H}_{i+1}(S^n - Y) \to \tilde{H}_{i+1}(S^n - R) \to \tilde{H}_i(S^n - X) \to \tilde{H}_i(S^n - e) \oplus \tilde{H}_i(S^n - Y) \to$$

$$\to \tilde{H}_i(S^n - R) \to \ldots$$

Note that e is homeomorphic to a tree, so by the result for trees, this sequence becomes simpler:

$$\ldots \to \tilde{H}_{i+1}(S^n - Y) \to \tilde{H}_{i+1}(S^n - R) \to \tilde{H}_i(S^n - X) \to \tilde{H}_i(S^n - Y) \to \tilde{H}_i(S^n - R) \to \ldots$$

Note that $R = h(v_0 \amalg v_1) \cong S^0$, so by proposition 2.B.1, $\tilde{H}_i(S^n - R)$ is isomorphic to $\mathbb{Z}$ if $i = n-1$ and 0 otherwise. So if $i \neq n-2$, then $\tilde{H}_{i+1}(S^n - R) = 0$. We also have $\tilde{H}_i(S^n - Y) = 0$ by the induction hypothesis. Therefore, if $i \neq n-2$, $\tilde{H}_i(S^n - X) = 0$.

When $i = n-2$,

$$\ldots \to \tilde{H}_{n-1}(S^n - Y) \to \tilde{H}_{n-1}(S^n - R) \to \tilde{H}_{n-2}(S^n - X) \to \tilde{H}_{n-2}(S^n - Y) \to \tilde{H}_{n-2}(S^n - R) \to \ldots$$

$\tilde{H}_{n-1}(S^n - Y) = 0$ and $\tilde{H}_{n-2}(S^n - Y) \cong \mathbb{Z}^m$ by induction hypothesis. But now $\tilde{H}_{n-1}(S^n - R) \cong \mathbb{Z}$, and $\tilde{H}_{n-2}(S^n - R) = 0$. So we have a short exact sequence:

$$0 \to \mathbb{Z} \to \tilde{H}_{n-2}(S^n - X) \to \mathbb{Z}^m \to 0$$

Since $\mathbb{Z}^m$ is free abelian, this sequence splits, hence $\tilde{H}_{n-2}(S^n - X) \cong \mathbb{Z}^{m+1}$. Therefore we have proved:

$$\tilde{H}_i(S^n - X) = \begin{cases} \mathbb{Z}^{m+1} & \text{if } i = n-2 \\ 0 & \text{otherwise} \end{cases}$$

which is isomorphic to $\tilde{H}_{n-i-1}(X) = \begin{cases} \mathbb{Z}^{m+1} & \text{if } n-i-1 = 1 \\ 0 & \text{otherwise} \end{cases}$

5. Let $U(2)$ denote the group of all unitary 2×2 matrices over $\mathbb{C}$ and $SU(2)$ the subgroup of $U(2)$ of unitary matrices with determinant one. We endow both sets with the topologies induced by the inclusions $SU(2) \subset U(2) \subset \mathbb{C}^4 \subset \mathbb{R}^8$.

(a) (5 marks) Show that $SU(2)$ is homeomorphic to S^3.

(b) (8 marks) Compute $H_*(U(2))$.

Solution

(a) Consider a 2×2 complex matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

X is an element of $SU(2)$ if and only if the following four conditions hold:

$$|a|^2 + |c|^2 = 1 \tag{1}$$

$$|b|^2 + |d|^2 = 1 \tag{2}$$

$$\overline{a}b + \overline{c}d = 0 \tag{3}$$

$$ad - bc = 1 \tag{4}$$

If we use (3) and (4) to eliminate the variable c, we obtain

$$a(|b|^2 + |d|^2) = \overline{d}$$

Applying (2), we find that $d = \overline{a}$. Similarly, if we use (1), (3), and (4) to eliminate d, we obtain $c = -\overline{b}$. So the elements of $SU(2)$ are those matrices of the form

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

such that $|a|^2 + |b|^2 = 1$. The map

$$\mathbb{C}^2 \to \{2 \times 2 \text{ matrices}\}$$
$$(a, b) \mapsto \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

is a homeomorphism onto its image, and so we see that $SU(2)$ can be identified with those vectors $(a, b) \in \mathbb{C}^2$ such that $|a|^2 + |b|^2 = 1$. In other words, the unit sphere in $\mathbb{C}^2 \approx \mathbb{R}^4$. This is the 3-sphere.

(b) A 2×2 matrix X belongs to $U(2)$ if and only if $XX^* = 1$, where X^* is the conjugate transpose. Thus if X is a unitary matrix, $(\det X)\overline{(\det X)} = 1$, or in other words $|\det X|^2 = 1$. So the determinant is a continuous map from $U(2)$ to the unit circle in $\mathbb{C}$. Let S^1 denote this unit circle.

The determinant is also a group homomorphism from $U(2)$ to the multiplicative group S^1. The preimage $\det^{-1}(1)$ is the kernel $SU(2)$. And for any $e^{i\lambda} \in S^1$, the preimage $\det^{-1}(e^{i\lambda})$ is the coset

$$\begin{pmatrix} e^{i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix} SU(2)$$

We may also write $e^{i\lambda/2}SU(2)$ to denote this coset. Multiplication by the scalar matrix $e^{-i\lambda/2}$ gives a homeomorphism between the coset and $SU(2)$.

So we have a map det to S^1 and the preimage of every point is homeomorphic to $SU(2) \approx S^3$. In fact, $U(2)$ is homeomorphic to $S^1 \times S^3$, but this fact is not obvious. Instead of proving it, we will apply the Mayer-Vietoris sequence.

Let $A \subset U(2)$ be the preimage under det of the set $\{e^{i\lambda} \in S^1 \mid -\pi < \lambda < \pi\}$, and let B be the preimage of $\{e^{i\lambda} \in S^1 \mid 0 < \lambda < 2\pi\}$. For every $X \in A$, there is a unique $\lambda \in (-\pi, \pi)$ such that $\det X = e^{i\lambda}$, and this λ varies continuously with X. So we can define

$$(X, t) \mapsto \begin{pmatrix} e^{-i\lambda t/2} & 0 \\ 0 & e^{-i\lambda t/2} \end{pmatrix} X$$

which gives us a deformation-retraction of A onto $SU(2)$. Similarly, for $X \in B$ there is a unique $\lambda \in (0, 2\pi)$ such that $\det X = e^{i\lambda}$, and we can define

$$(X,t) \mapsto \begin{pmatrix} e^{it(\pi-\lambda)/2} & 0 \\ 0 & e^{it(\pi-\lambda)/2} \end{pmatrix} X$$

which gives us a deformation-retraction of B onto the coset $-SU(2)$.

The intersection $A \cap B$ has two components: Let C be the component $\det^{-1}(\{e^{i\lambda} \in S^1 \mid 0 < \lambda < \pi\})$ and let D be the component $\det^{-1}(\{e^{i\lambda} \in S^1 \mid \pi < \lambda < 2\pi\})$. Just as with A and B, one can show that C deformation-retracts onto the coset $iSU(2)$ and D deformation-retracts onto the coset $-iSU(2)$.

Now we have the following maps, all of which are isomorphisms:

$$\begin{tikzcd}
 & & H_3(iSU(2)) \arrow[d] & & \\
 & & H_3(C) \arrow[dl] \arrow[dr] & & \\
H_3(-SU(2)) \arrow[r] & H_3(B) & & H_3(A) & H_3(SU(2)) \arrow[l] \\
 & & H_3(D) \arrow[ul] \arrow[ur] & & \\
 & & H_3(-iSU(2)) \arrow[u] & &
\end{tikzcd}$$

(The diagram might not commute.) In order to apply Mayer-Vietoris, we will need to know the degrees of these maps. First, let us choose a generator of $H_3(SU(2)) \approx \mathbb{Z}$. Then let us choose the unique generators for the other groups such that every map in the diagram except possibly the lower right diagonal map takes a chosen generator to a chosen generator:

$$\begin{tikzcd}
 & & H_3(iSU(2)) \arrow[d, "\mathbb{1}"] & & \\
 & & H_3(C) \arrow[dl, "\mathbb{1}"] \arrow[dr, "\mathbb{1}"] & & \\
H_3(-SU(2)) \arrow[r, "\mathbb{1}"] & H_3(B) & & H_3(A) & H_3(SU(2)) \arrow[l, "\mathbb{1}"'] \\
 & & H_3(D) \arrow[ul, "\mathbb{1}"] \arrow[ur] & & \\
 & & H_3(-iSU(2)) \arrow[u, "\mathbb{1}"] & &
\end{tikzcd}$$

Now let's consider two homeomorphisms $SU(2) \to -iSU(2)$. We have the map f which is multiplication by the scalar matrix $e^{i3\pi/4}$; this induces the composition of isomorphisms going counterclockwise around the diagram, and it is degree 1. We also have the map g which is multiplication by the scalar matrix $e^{-i\pi/4}$; this induces the composition of isomorphisms going clockwise around the diagram. So $g = e^{i3\pi/4}e^{-i\pi} = f \circ -\mathbb{1}$, the composition of f with the antipodal map on $SU(2) \approx S^3$. The antipodal map on S^3 is degree 1, so g is degree 1 also.

Now we're ready for the Mayer-Vietoris sequence. One bit of it looks like this:

$$0 \to H_4(U(2)) \to H_3(A \cap B) \to H_3(A) \oplus H_3(B) \to H_3(U(2)) \to 0$$

The map in the middle is

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

The map is of rank 1 and so the kernel and cokernel are both $\mathbb{Z}$. The other interesting bit of the (reduced) sequence looks like

$$0 \to H_1(U(2)) \to \widetilde{H}_0(A \cap B) \to 0$$

so $H_1(U(2)) \approx \widetilde{H}_0(A \cap B) \approx \mathbb{Z}$.

$$H_i(U(2)) = \begin{cases} \mathbb{Z} & \text{for} \qquad i = 0, 1, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$