MATH 215B. SOLUTIONS TO HOMEWORK 3

1. (6 marks) Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]

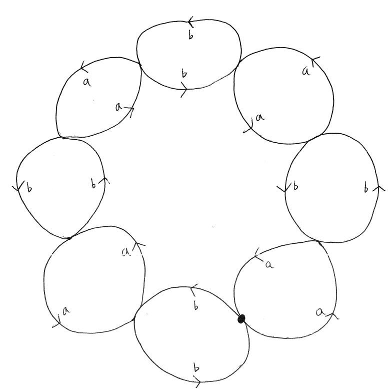
Solution

Call the map in question f. $\pi_1(X)$ is finite, so the image of f_* is finite. The only finite subgroup of $\pi_1(S^1) \approx \mathbb{Z}$ is 0. So the lifting criterion, Proposition 1.33, tells us that there's a lift $\widetilde{f} : X \to \mathbb{R}$ to the universal cover $\mathbb{R} \to S^1$. Now $\mathbb{R}$ is contractible, so there's a nullhomotopy of $\widetilde{f}$; that is, a map $h : X \times I \to \mathbb{R}$ such that $h_0 = \widetilde{f}$ and h_1 is a constant map. Composing this nullhomotopy with the covering map $p : \mathbb{R} \to S^1$ gives a nullhomotopy of f; that is, $p \circ h : X \times I \to S^1$ satisfies $p \circ h_0 = f$ and $p \circ h_1$ is a constant.

If you're worried because Proposition 1.33 is about based maps, choose a basepoint x_0 in X. Then f is a based map $(X, x_0) \to (S^1, f(x_0))$. We can choose a basepoint $r_0 \in \mathbb{R}$ such that $p(r_0) = f(x_0)$. Then Proposition 1.33 guarantees us a based lift $\widetilde{f}$ such that $\widetilde{f}(x_0) = r_0$. $\mathbb{R}$ deformation-retracts onto r_0, so $\widetilde{f}$ is homotopic to the map with constant value r_0. Composing with p gives a based homotopy from f to the map with constant value $f(x_0)$.

2. (10 marks) Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

Solution

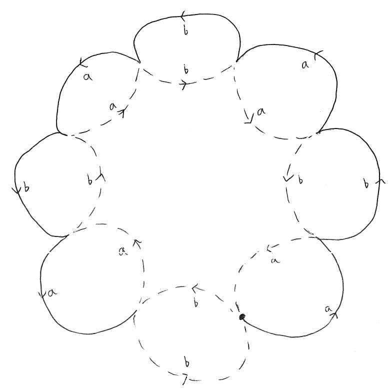


The edges labelled a (respectively b) are mapped homeomorphically onto the circle a (respectively b), as on page 58 in Hatcher.

Let us call this covering space X, and the covering map p. There are two circles adjacent to the basepoint, and loops tracing these circles are mapped to a^2 and b^2 in $\pi_1(S^1 \vee S^1)$. A loop tracing out the outer edge of the graph maps to $(ab)^4$ in $\pi_1(S^1 \vee S^1)$. Also, X is a normal covering space, so $p_*\pi_1(X)$ must be normal.

We now know that $p_*\pi_1(X)$ contains the normal subgroup generated by a^2, b^2, and $(ab)^4$. It now remains to show that $p_*\pi_1(X)$ is in fact equal to that normal subgroup. Since p_* is injective, it suffices to show that $\pi_1(X)$ is generated by conjugates of a^2, b^2, and $(ab)^4$.

Proposition 1A.2 in Hatcher tells us how to calculate $\pi_1(X)$ using a maximal tree:



Using that method, we see that $\pi_1(X)$ is generated by the following nine elements:

$$a^2$$
$$ab^2a^{-1}$$
$$(ab)a^2(ab)^{-1}$$
$$(aba)b^2(aba)^{-1}$$
$$(abab)a^2(abab)^{-1}$$
$$(ababa)b^2(ababa)^{-1}$$
$$(ababab)a^2(ababab)^{-1}$$
$$(ab)^4$$
$$abababab^{-1} = (ab)^4b^{-2}$$

which are all conjugates or products of a^2, b^2, and $(ab)^4$.

3. (12 marks) Let $U_+ = \{z \in S^1 \mid Im(z) > 0\}$ and $U_- = \{z \in S^1 \mid Im(z) < 0\}$. Let $\sim$ be the equivalence relation on S^1 where $x \sim y$ if and only if either ($x \in U_+$ and $y \in U_+$) or ($x \in U_-$ and $y \in U_-$) or $x = y$. Let $X = S^1/\sim = \{[1], [-1], U_+, U_-\}$ have the quotient topology. Let $q : S^1 \to X$ be the quotient map. Prove that $q_* : \pi_1(S^1, 1) \to \pi_1(X, [1])$ is an isomorphism.

Solution

First of all, note that there are only a few open sets in X. U_+ and U_- are open points. The smallest neighborhood of [1] is $E_1 := \{U_+, [1], U_-\}$, and the smallest neighborhood of $[-1]$ is $E_{-1} := \{U_+, [-1], U_-\}$.

Consider the covering map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. Let $U_+^n = (n, n + \frac{1}{2})$ and $U_-^n = (n + \frac{1}{2}, n + 1)$. Say $x \sim y$ iff ($x, y \in U_+^n$ for some n), or ($x, y \in U_-^n$ for some n), or $x = y$. Let $\widetilde{X} = \mathbb{R}/\sim$ with the quotient topology. Let E^n denote $\{U_-^{n-1}, [n], U_+^n\}$, the smallest neighborhood of $[n]$, and let $E^{n+\frac{1}{2}}$ denote $\{U_+^n, [n + \frac{1}{2}], U_-^n\}$, the smallest neighborhood of $[n + \frac{1}{2}]$.

$qp : \mathbb{R} \to X$ factors through a map $\widetilde{X} \to X$ which we will also call p; it takes $[n]$ to $[1]$, $[n+\frac{1}{2}]$ to $[-1]$, U_+^n to U_+, and U_-^n to U_-. It is easy to check that it is continuous:

$$\begin{aligned} p^{-1}(U_+) &= \cup_n U_+^n \\ p^{-1}(U_-) &= \cup_n U_-^n \\ p^{-1}(E_1) &= \cup_n E^n \\ p^{-1}(E_{-1}) &= \cup_n E^{n+\frac{1}{2}} \end{aligned}$$

In fact, this shows that p is a covering map.

In order to apply Proposition 1.39 in Hatcher, we must prove a number of properties of X and $\widetilde{X}$:

X is path-connected: There is an arc between any two points in S^1. Identifying the arc with the interval and composing with q gives continuous paths in X between any two points.

X is locally path-connected: There is an arc between any two points in $S^1 - \{-1\}$, and the arc can be chosen to lie entirely in $S^1 - \{-1\}$. After identifying the arc with the interval and composing with q, we get a path between any two points in E_1. By a similar argument, E_{-1} is path-connected. The open points U_+ and U_- are of course path-connected.

$\widetilde{X}$ is path-connected: There is a segment connecting any two points in $\mathbb{R}$. By the same reasoning as in the above two paragraphs, $\widetilde{X}$ is path-connected.

$\widetilde{X}$ is simply-connected: We will show that it is in fact contractible. Consider the following deformation-retraction of $\mathbb{R}$:

$$\begin{aligned} \mathbb{R} \times I \to & \mathbb{R} \\ (x,t) \mapsto & \begin{cases} x & \text{if} \quad t = 0 \\ x & \text{if} \quad |x| \le \frac{1}{t} - 1 < \infty \\ \frac{1}{t} - 1 & \text{if} \quad \frac{1}{t} - 1 \le x \\ 1 - \frac{1}{t} & \text{if} \quad x \le 1 - \frac{1}{t} \end{cases} \end{aligned}$$

At each time t it retracts $\mathbb{R}$ onto the interval $[1-\frac{1}{t}, \frac{1}{t}-1]$ by collapsing everything outside of that interval onto the endpoints. Note that the image of each $U_\pm^n \times \{t\}$ is contained in some $U_\pm^m$ or some $\{m\}$ or some $\{m+\frac{1}{2}\}$, depending on t. Thus this deformation-retraction descends to a deformation-retraction of $\widetilde{X}$.

Thus $\widetilde{X}$ is the universal cover of X, and by Proposition 1.39, $\pi_1(X)$ is isomorphic to the group of deck transformations. The deck transformations of $\mathbb{R} \to S^1$ descend to the deck transformations of $\widetilde{X} \to X$, so we have $\pi_1(X) \approx \mathbb{Z}$.

Furthermore, we want to know if q_* is an isomorphism. Equivalently, we want to know whether the loop q represents a generator of $\pi_1(X)$. The loop q lifts to a path in $\widetilde{X}$ from $[0]$ to $[1]$; and the deck transformation that takes $[0]$ to $[1]$ generates the group of deck transformations. So $[q]$ is a generator of $\pi_1(X)$.

4. (12 marks) For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\widetilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X and that such a 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$.

Solution

Recall that for every group G, the commutator subgroup $[G, G]$ is the subgroup generated by elements of the form $ghg^{-1}h^{-1}$, for $g, h \in G$. The quotient $G/[G, G]$ is the abelianization G_{ab}. It satisfies the following universal property: If A is an abelian group and $f : G \to A$ is a homomorphism, then f factors through G_{ab}; that is, there is a map $\widetilde{f} : G_{\text{ab}} \to A$ such that $G \to G_{\text{ab}} \xrightarrow{\widetilde{f}} A$ is the same as f.

By the classification theorem for covering spaces, the commutator subgroup $[\pi_1(X), \pi_1(X)]$ determines a path-connected covering space $\widetilde{X} \xrightarrow{p} X$. Since the commutator subgroup is normal, $\widetilde{X}$ is a normal covering space. And so the group of deck transformations $G(\widetilde{X})$ is isomorphic to $\pi_1(X)/[\pi_1(X), \pi_1(X)] = \pi_1(X)_{\text{ab}}$, which is abelian. So $\widetilde{X}$ is an abelian covering space. Let's give it a basepoint, too.

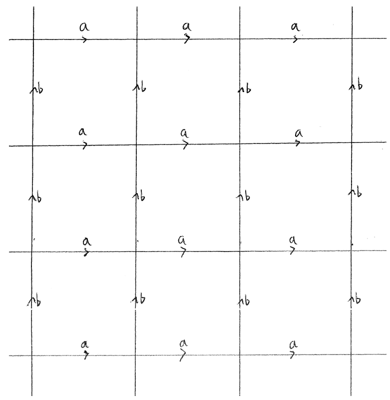
Suppose we have another abelian covering space $q : Y \to X$. This means that $q_*\pi_1(Y)$ is a normal subgroup of $\pi_1(X)$ and the quotient $\pi_1(X)/q_*\pi_1(Y)$ is abelian. By the universal property of abelianization, the quotient map $\pi_1(X) \to \pi_1(X)/q_*\pi_1(Y)$ factors through $\pi_1(X)/[\pi_1(X), \pi_1(X)]$. This means that the commutator subgroup of $\pi_1(X)$ lies inside the subgroup $q_*\pi_1(Y)$. So by the lifting criterion of covering maps, the map $p : \widetilde{X} \to X$ lifts to a (based) map $\widetilde{p} : \widetilde{X} \to Y$ such that $q\widetilde{p} = p$.

$\widetilde{p}$ is the lift of a covering map over a covering map, and so is itself a covering map. To see this, suppose $y \in Y$. We can choose a neighborhood $U \subset X$ of $q(y)$. Then $q^{-1}(U)$ is a neighborhood of y that is a disjoint union of open sets that are mapped homeomorphically to U. Then $\widetilde{p}^{-1}\big(q^{-1}(U)\big) = p^{-1}(U)$, which is a disjoint union of open sets that are mapped homeomorphically by $q\widetilde{p}$ to U. Equivalently, it is a disjoint union of open sets that are mapped homeomorphically by $\widetilde{p}$ to $q^{-1}(U)$. Thus $\widetilde{p}$ is a covering map onto Y.

Thus we see that $\widetilde{X}$ is a universal abelian covering space, in the sense that it covers every abelian covering space of X. Now we will show that abelian covering spaces having such a universal property are unique. For suppose that $q : Y \to X$ is also a universal abelian covering space. By universality of $\widetilde{X}$, we have a (based) covering map $\widetilde{p} : \widetilde{X} \to Y$ satisfying $q\widetilde{p} = p$, and by universality of Y we have a (based) covering map $\widetilde{q} : Y \to \widetilde{X}$ satisfying $p\widetilde{q} = q$. Thus $p\widetilde{q}\widetilde{p} = q\widetilde{p} = p$.

Now we use the unique lifting property (Proposition 1.34 in Hatcher). Both $\widetilde{q}\widetilde{p}$ and $\mathbb{1}_{\widetilde{X}}$ are lifts of p over p. That is, $p\widetilde{q}\widetilde{p} = p\mathbb{1}_{\widetilde{X}} = p$. And they both carry the basepoint to the basepoint. So therefore $\widetilde{q}\widetilde{p}$ and $\mathbb{1}_{\widetilde{X}}$ are the same. By a similar argument we have $\widetilde{p}\widetilde{q} = \mathbb{1}_Y$. Thus $\widetilde{p}$ and $\widetilde{q}$ are inverse isomorphisms. Thus the universal abelian cover is unique up to isomorphism.

Now let's take $X = S^1 \vee S^1$. We're looking for a normal covering space whose group of deck transformations is $\pi_1(S^1 \vee S^1)_{\text{ab}} \approx \mathbb{Z} \times \mathbb{Z}$. It is easy to see that this infinite planar graph is such a covering space:



Every word $g \in \pi_1(S^1 \vee S^1) \approx \mathbb{Z} * \mathbb{Z}$ corresponds to a path in $\widetilde{X}$ that is unique once we choose a starting point. For example, the word ab^2a^{-1} corresponds to a path going right one unit, up two units, then left one unit. If g and h are two words in $\pi_1(S^1 \vee S^1)$, then $ghg^{-1}h^{-1}$ corresponds to a loop in $\widetilde{X}$. It represents a homotopy class whose image in $\pi_1(S^1 \vee S^1)$ is $[ghg^{-1}h^{-1}]$. Thus $p_*\pi_1(\widetilde{X})$ contains the commutator subgroup. If it contained the commutator subgroup properly, then $G(\widetilde{X})$ would be a proper quotient of $\mathrm{ab}\,\pi_1(S^1 \vee S^1)$; that is, there would be a surjective but not bijective homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. But this cannot be. So $p_*\pi_1(\widetilde{X})$ is exactly the commutator subgroup, and $\widetilde{X}$ is indeed the universal abelian cover.

5. A section of a covering space $p : E \to X$ is a continuous map $s : X \to E$ such that $ps(x) = x$ for all $x \in X$. Let $p : E \to X$ be a covering space over a path connected, locally path connected and semi-locally simply connected X and let $x_0 \in X$.

(a) (6 marks) Show that there is a bijection between the set of sections of p and the set of $y \in p^{-1}(x_0)$ such that any lift of a loop based at x_0 that starts at y must be a loop.

(b) (4 marks) Given a positive integer $n \geq 2$, deduce that there are no continuous maps $f : S^1 \to S^1$ such that $f(x)^n = x$ for all x, where $f(x)^n$ is the nth power of $f(x)$ as a complex number.

Solution

(a) Let A be the set of sections of p and B the set of $y \in p^{-1}(x_0)$ such that any lift of a loop based at x_0 that starts at y must be a loop. Now define $\phi : A \to B$ by $\phi(s) = s(x_0)$.

This map is well defined. If s is a section, $ps(x_0) = x_0$, so $s(x_0) \in p^{-1}(x_0)$. Also, given a loop γ based at x_0, the composition $\alpha = s\gamma$ is a lift of γ to E with $\alpha(0) = s(x_0)$. And α is a loop. It is also the only lift of γ starting at $s(x_0)$ by uniqueness of lifts.

ϕ is injective. Let s, r be two sections of p such that $s(x_0) = r(x_0)$. Note that a section of p is a lift of the identity $1_X : X \to X$ to E. Since X is path connected, by the unique lifting property, we must have $s = r$.

ϕ is surjective. Let $y \in B$. We are looking for a lift s of $1_X : X \to X$ that takes x_0 to y. Since X is path connected and locally path connected, the lifting criterion says that such a lift exists if and only if $\pi_1(X, x_0) \leq p_*(\pi_1(E, y))$. Now let γ be a loop in X based at x_0, and take a lift α in E starting at y. Because $y \in B$, α is a loop and therefore we can consider $[\alpha] \in \pi_1(E, y)$. Note that $p_*([\alpha]) = [\gamma]$, so $\pi_1(X, x_0) \leq p_*(\pi_1(E, y))$ and therefore one such lift exists.

(b) Consider the covering space $p : S^1 \to S^1$ that takes x to x^n, where this denotes the nth power of x as a complex number. Assume there existed a map $f : S^1 \to S^1$ with $f(x)^n = x$ for all x. Then $pf(x) = x$, that is, f is a section of p.

Take the basepoint $1 \in S^1$. By part (a), there is a bijection between the set of sections and the set of $y \in p^{-1}(1)$ such that any lift of a loop based at 1 that starts at y is a loop. Consider the loop $f : I \to S^1$ given by $f(t) = e^{2\pi i t}$. The class $[f]$ represents the generator in $\pi_1(S^1, 1)$. If a lift of f starting at y is a loop, then $[f] \in p_*(\pi_1(S^1, y))$. But $p_*(\pi_1(S^1, y))$ is the subgroup $n\mathbb{Z}$ of $\mathbb{Z} \cong \pi_1(S^1, 1)$. If $n \geq 2$, this subgroup does not contain the generator of $\mathbb{Z}$, giving a contradiction.

Alternatively, you can think of the loop f as the identity map of S^1. By the lifting criterion, there is a lift to the cover if and only if $\pi_1(S^1, 1) \leq p * (\pi_1(S^1, y))$, and again you reach a contradiction if $n \geq 2$ because $\mathbb{Z}$ is not contained in $n\mathbb{Z}$.

6. For a finite graph X define the Euler characteristic $\chi(X)$ to be the number of vertices minus the number of edges.

(a) (7 marks) Show that $\chi(X) = 1$ if X is a tree, and that the rank of $\pi_1(X)$ (the number of elements in a basis) is $1 - \chi(X)$ is X is connected. Deduce that the Euler characteristic is a homotopy invariant of connected graphs.

(b) (3 marks) Let n and k be positive integers, and let F be the free group on n generators. Show that if G is an index k subgroup of F, then G is a free group on $kn - k + 1$ generators.

Solution

(a) A tree is a contractible graph. First we want to know that every finite tree X has a leaf vertex. Following the proof of Proposition 1A.1, we let X_0 be a trivial tree consisting of one vertex of X, and then we inductively define X_{i+1} by adjoining to X_i the closures of all the edges that touch X_i. Since X is finite, this process terminates, say at stage n. Choose a vertex $a \in X_n - X_{n-1}$. If there were two edges in $X_n - X_{n-1}$ joining a to X_{n-1}, then if we combined those with a non-self-intersecting path in X_{n-1} we would get a cycle, and so X would not be simply-connected, let alone contractible. So a is joined to the rest of X by just one edge. (Either that or $n = 0$ and X is a point.) This is our leaf vertex.

Now suppose X is a finite tree, so either it has only one point or it has a leaf vertex. In the latter case, we can remove the leaf node vertex and the edge connected to it without changing the Euler characteristic. Since the tree is finite, iterating this process will result in the trivial tree with only one point. The Euler characteristic of the trivial tree is 1, so the Euler characteristic of X is 1.

Now suppose X is a finite connected graph. It has a maximal tree T, and by Proposition 1A.2 in Hatcher, $\pi_1(X)$ is a free group with rank equal to the number of edges comprising $X - T$. This number is equal to $\chi(T) - \chi(X) = 1 - \chi(X)$.

Thus $\chi(X)$ is uniquely determined by the free rank of $\pi_1(X)$. Since π_1 is a homotopy invariant, so is χ.

(b) Let $X = \vee_n S^1$. Then $\pi_1(X) \approx F$. G can be considered to be an index k subgroup of $\pi_1(X)$, so there is a k-sheeted covering space $p : \widetilde{X} \to X$ such that $p_*\pi_1(\widetilde{X}) \approx G$.

$\widetilde{X}$ is a covering space of a graph, so is a graph itself. It is k-sheeted, so it has k things for every thing X has: k vertices and kn edges. So $\chi(\widetilde{X}) = k - kn$.

Now $G \approx p_*\pi_1(\widetilde{X}) \approx \pi_1(\widetilde{X})$. By (a) this is free with rank $1 - \chi(\widetilde{X}) = 1 - (k - kn) = kn - k + 1$.