

MATH 215B. SOLUTIONS TO HOMEWORK 2

1. (6 marks) Construct a path connected space X such that $\pi_1(X, x_0) \cong D_4$, the dihedral group with 8 elements.

Solution A presentation of D_4 is $\langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$, where a is a counterclockwise rotation by 90 degrees of the square and b is the reflection across the middle horizontal, bisecting line of the square.

Following Corollary 1.28, X can be constructed as follows: Start with $S^1 \vee S^1$, whose fundamental group is free on two generators. Now take loops a and b representing these generators. Now attach three 2-cells $e_{a^4}^2, e_{b^2}^2, e_{(ab)^2}^2$ with attaching maps given by the loops $a^4, b^2, (ab)^2$, respectively. The resulting space X has fundamental group $\langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$. To be more precise, $X = S^1 \vee S^1 \amalg_{a^4} D^2 \amalg_{b^2} D^2 \amalg_{(ab)^2} D^2$.

2. (8 marks) Suppose a space Y is obtained from a path-connected subspace X by attaching n -cells for a fixed $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism on π_1 . [See the proof of Proposition 1.26]

Solution Let $x_0 \in X$ and $s_0 \in S^{n-1}$. Assume $Y = X \amalg_{\amalg_{\alpha \in J} f_\alpha} \amalg_{\alpha \in J} D_\alpha^n$, where $D_\alpha^n = D^n$ for all α , J is some index set and $f_\alpha : S_\alpha^{n-1} = \partial D_\alpha^n \rightarrow X$. Pick a path γ_α from x_0 to $f_\alpha(s_0)$.

Construct a space Z by attaching rectangular strips $R_\alpha = I \times I$ to Y in the following way. Identify the lower edge $I \times \{0\}$ with $\gamma_\alpha(I)$, the right edge $\{1\} \times I$ with an arc in D_α^n and the left edges $\{0\} \times I$ with all the left edges of the other rectangular strips. Z deformation retracts onto Y just by pushing the second coordinate of each strip to zero.

In each cell D_α^n , choose a point y_α not in the arc along which R_α is attached. Let $A = Z - \bigcup_{\alpha \in J} \{y_\alpha\}$ and let $B = Z - X$. Note that A , B and $A \cap B$ are path connected and open. Then A deformation retracts onto X , and B is contractible. Since $\pi_1(B) = 0$, we obtain by van Kampen that $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(A)$ by the normal subgroup generated by the image of the map $\pi_1(A \cap B) \rightarrow \pi_1(A)$.

We claim that $\pi_1(A \cap B)$ is trivial. To see this, use van Kampen's theorem again with the open cover $A_\alpha = A \cap B - \bigcup_{\beta \neq \alpha} D_\beta^n$. Again, the A_α 's are open and path connected, the intersection of all of them is not empty and double and triple intersections are path connected. Note that A_α deformation retracts onto S_α^{n-1} , so $\pi_1(A_\alpha) = \pi_1(S^{n-1}) = 0$ (since $n \geq 3$). Therefore, $\pi_1(A \cap B)$ is trivial and so $\pi_1(Y) \cong \pi_1(Z) \cong \pi_1(A) \cong \pi_1(X)$.

Aside comment. Note that the technicality of attaching the rectangular strips comes up only because we are attaching more than one cell at a time and we need open path connected subsets to apply van Kampen's theorem. If we were only

attaching one cell, we could pick a basepoint x_0 in the interior of the cell and take $A = Y - \{y_1\}$ and $B = Y - X$, where y_1 is a point in the interior of the cell different from x_0 .

3. (12 marks) In the surface M_g of genus g , let C be a circle that separates M_g into two compact subsurfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C , and hence M_g does not retract onto C . [Hint: abelianize π_1 .] But show that M_g does retract onto the nonseparating circle C' in the figure.

Solution

Let us suppose that there is a retraction $r : M'_h \rightarrow C$. In particular, this means that the composition $\pi_1(C) \xrightarrow{i_*} \pi_1(M'_h) \xrightarrow{r_*} \pi_1(C)$ is the identity. Abelianizing the groups, we get maps $\pi_1(C) \xrightarrow{i_*} \text{ab}(\pi_1(M'_h)) \xrightarrow{r_*} \pi_1(C)$. ($\pi_1(C)$ is already abelian.) In particular, this implies that $\pi_1(C) \xrightarrow{i_*} \pi_1(M'_h) \rightarrow \text{ab}(\pi_1(M'_h))$ is injective. We will obtain a contradiction.

The closed surface M_h has a cell structure consisting of one 0-cell, $2h$ 1-cells, and one 2-cell. The surface-with-boundary M'_h is M_h minus an open 2-disk B . In choosing a cell structure on M_h , we can ensure that \overline{B} lies in the interior of the 2-cell.

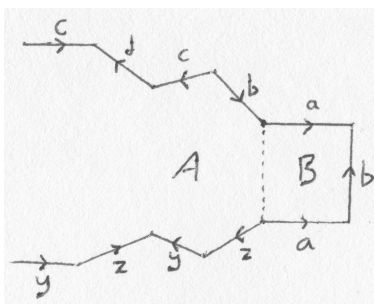
There is a slight inconvenience posed by the fact that the basepoint c_0 of C is not the 0-cell of M'_h (which we might call m_0). So choose a path γ in M'_h from m_0 to c_0 , whose interior lies in the 2-cell; this determines an isomorphism $\pi_1(M'_h, m_0) \approx \pi_1(M'_h, c_0)$. A generator $[g]$ of $\pi_1(C, c_0)$ is mapped under inclusion to a class $[g]$ in $\pi_1(M'_h, c_0)$, and corresponds to a class $[\gamma \cdot g \cdot \overline{\gamma}]$ in $\pi_1(M'_h, m_0)$.

If we think of the 2-cell as the interior of a closed disk D^2 , then we see that the D^2 minus B deformation-retracts onto the boundary of D^2 . In particular, $\gamma \cdot g \cdot \overline{\gamma}$ is homotopic in the complement of B to a path tracing out ∂D^2 . Composing this with the characteristic map $D^2 \rightarrow M_h$, we get a homotopy of loops in M'_h from $\gamma \cdot g \cdot \overline{\gamma}$ to a loop tracing out the attaching map of the 2-cell. Its homotopy class is the product of commutators $[a_1, b_1] \cdots [a_h, b_h]$ (or the inverse of that class, depending on which generator of $\pi_1(C)$ we started with). Here a_i and b_i are paths tracing out the 1-cells.

Once we abelianize $\pi_1(M'_h)$, the commutators vanish, and so the composition $\pi_1(C, c_0) \rightarrow \pi_1(M'_h, c_0) \approx \pi_1(M'_h, m_0) \rightarrow \text{ab}(\pi_1(M'_h, m_0))$ is trivial. So M'_h cannot retract onto C . So M_g cannot retract onto C , because such a retraction would restrict to a retraction of M'_h onto C .

However, M_g does retract onto C' : We can construct M_g from a $4g$ -gon by identifying pairs of edges. We label the sides a, b, \dots . We can identify C' with a . The dotted line in the figure partitions the polygon into two regions, A and B . B is a square, and vertical projection gives a retraction of B onto the lower side labeled a which identifies the two sides labeled a in the manner indicated by the arrows. The vertical projection also takes the right side to the lower right corner and the dotted line to the lower left corner. We can extend this retraction to the entire polygon by sending A to the lower-left corner. To check that this induces a retraction of M_g onto a , we must check that it respects the identifications of pairs of edges. Indeed, all edges not labeled a or b get sent to the lower left corner; and

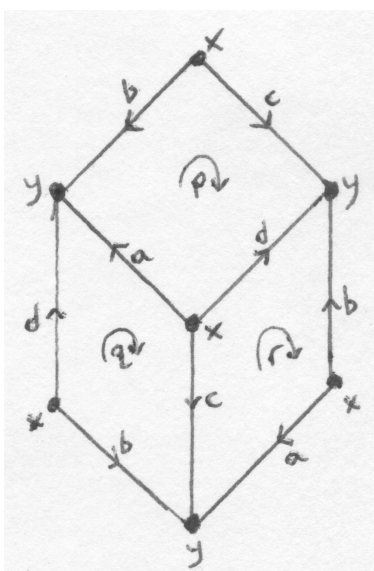
the two edges labeled b get sent to corners, which are identified in M_g ; and we already noted that the retraction identifies the two edges labeled a .



4. (10 marks) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show that this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

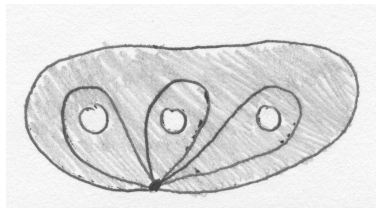
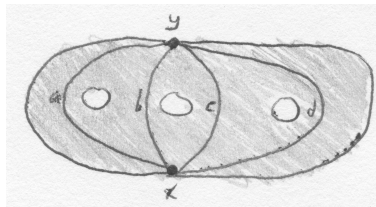
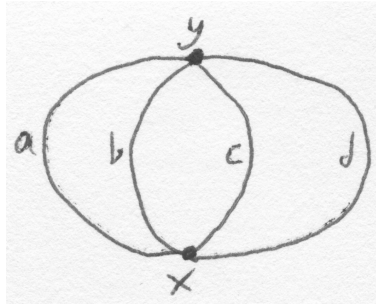
Solution

The cube, being a polyhedron, has a familiar cell complex with eight 0-cells, twelve 1-cells, six 2-cells, and one 3-cell. The identification leaves the 3-cell alone, and identifies pairs of 2-cells. So the cell complex of the quotient space has just three 2-cells and one 3-cell, which are sketched below. The 2-cells are labeled p , q , and r .



What it does to the 1-cells and 0-cells is more complicated: Successive screw actions may identify multiple cells together. But the screw action always takes a cell homeomorphically onto another cell, so the quotient space is a cell complex. It turns out that there are four distinct 1-cells (labeled a , b , c , and d) and two distinct 0-cells (labeled x and y). (The arrows are not part of the cell structure.)

The 1-skeleton X^1 is homotopy-equivalent to a wedge of three circles. To see this, let M be a closed 2-disk with three disks removed from its interior. We can embed either X^1 or $\bigvee_3 S^1$ into M so that M deformation-retracts onto it; therefore X^1 and $\bigvee_3 S^1$ are homotopy-equivalent to M , and so they are homotopy-equivalent to each other. Therefore $\pi_1(X^1) \approx *_3\mathbb{Z}$. For generators we can choose the paths $a \cdot \bar{b}$, $a \cdot \bar{c}$, and $a \cdot \bar{d}$. Let i , j , and k denote these loops' respective homotopy classes. So we can give a presentation $\pi_1(X^1) = \langle i, j, k \rangle$.



Now we use Proposition 1.26 to compute $\pi_1(X^2)$. X^2 is obtained from X^1 by attaching three 2-cells. The attaching maps can be thought of as loops. As a loop, the attaching map for p is equal to $a \cdot \bar{b} \cdot c \cdot \bar{d}$. The homotopy class in X^1 is

$$\begin{aligned} [a \cdot \bar{b} \cdot c \cdot \bar{d}] &= [a \cdot \bar{b} \cdot c \cdot \bar{a} \cdot a \cdot \bar{d}] \\ &= [(a \cdot \bar{b}) \cdot (\bar{a} \cdot c) \cdot (a \cdot \bar{d})] \\ &= ij^{-1}k \end{aligned}$$

Similarly, we compute the homotopy class of the attaching map for q (thought of as a loop):

$$\begin{aligned} [c \cdot \bar{b} \cdot d \cdot \bar{a}] &= [c \cdot \bar{a} \cdot a \cdot \bar{b} \cdot d \cdot \bar{a}] \\ &= [\overline{(a \cdot \bar{c})} \cdot (a \cdot \bar{b}) \cdot \overline{(a \cdot \bar{d})}] \\ &= j^{-1}ik^{-1} \end{aligned}$$

And the homotopy class of the attaching map for r :

$$\begin{aligned} [d \cdot \bar{b} \cdot a \cdot \bar{c}] &= [d \cdot \bar{a} \cdot a \cdot \bar{b} \cdot a \cdot \bar{c}] \\ &= [\overline{(a \cdot \bar{d})} \cdot (a \cdot \bar{b}) \cdot (a \cdot \bar{c})] \\ &= k^{-1}ij \end{aligned}$$

By Proposition 1.26, $\pi_1(X^2) = \langle i, j, k \mid ij^{-1}k = 1, j^{-1}ik^{-1} = 1, k^{-1}ij = 1 \rangle = \langle i, j, k \mid j = ki, i = jk, ij = k \rangle$.

X is obtained from X^2 by attaching a 3-cell. In general, attaching a cell of dimension ≥ 3 does not affect the fundamental group. (See Exercise 6 in Section 1.2.) But it's easy to use van Kampen's theorem to prove in this case that $\pi_1(X) \approx \pi_1(X^2)$: Let β and σ denote two distinct points in the interior of the 3-cell, and take β to be our basepoint. Let $A = X - \{\sigma\}$ and let B be the interior of the 3-cell. Then $X = A \cup B$, and we apply van Kampen. A deformation-retracts onto X^2 , and B is contractible. $A \cap B$ is the interior of the 3-cell minus a point, and is homotopy-equivalent to S^2 . Van Kampen tells us that $\pi_1(X) \approx \pi_1(X^2)$.

Now the group $\langle i, j, k \mid j = ki, i = jk, ij = k \rangle$ is actually the quaternion group. We can derive the relations:

$$\begin{aligned} j^2 &= (ik^{-1})(ki) = i^2 \\ k^2 &= (ij)(j^{-1}i) = i^2 \end{aligned}$$

and we can let -1 denote the element $i^2 = j^2 = k^2$. Then we can deduce:

$$(-1)^2 = i^2j^2 = (k^{-1}k)i(k)j = k^{-1}(j)kj = k^{-1}(i)j = k^{-1}k = 1$$

From here it is not difficult to reconstruct the multiplication table for the quaternion group.

5. (12 marks) Given a space X with basepoint $x_0 \in X$, we may construct a CW complex $L(X)$ having a single 0-cell, a 1-cell e_γ^1 for each loop in γ in X based at x_0 , and a 2-cell e_τ^2 for each map τ of a standard triangle PQR into X taking the three vertices P , Q , and R of the triangle to x_0 . The 2-cell e_τ^2 is attached to the three 1-cells that are the loops obtained by restricting τ to the three oriented edges PQ , PR , and QR . Show that the natural map $L(X) \rightarrow X$ induces an isomorphism $\pi_1(L(X)) \approx \pi_1(X, x_0)$. [Rephrased: Given any space X , there is a 2-dimensional CW complex equipped with a map to X that induces isomorphism in the fundamental group. This is stronger than Corollary 1.28.]

Solution

The “natural map” $f : L(X) \rightarrow X$ in question takes the 0-cell $*$ to x_0 ; if we think of e_γ^1 as $(0, 1)$ and t is a point in e_γ^1 , then $f(t) = \gamma(t)$; and if we identify the cell e_τ^2 with the triangle PQR , and $t \in e_\tau^2$, then $f(t) = \tau(t)$.

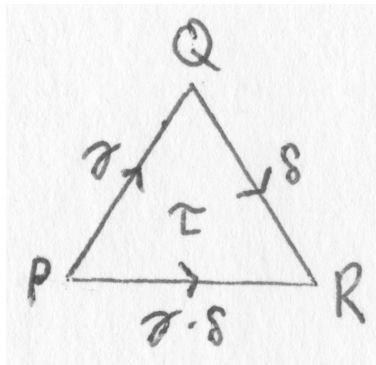
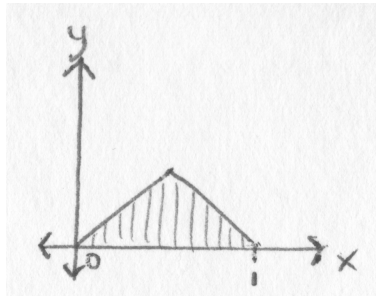
For every loop γ in (X, x_0) , there is a corresponding 1-cell e_γ^1 in $L(X)$. Let p_γ denote the path in $L(X)$ that traces out e_γ^1 . Note that $f \circ p_\gamma = \gamma$.

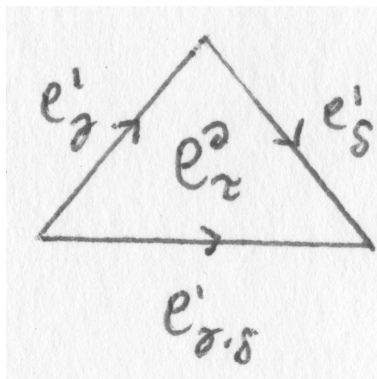
First we prove surjectivity of f_* . Suppose γ is a loop in X based at x_0 . Then $f_*[p_\gamma] = [f \circ p_\gamma] = [\gamma]$.

Injectivity is more difficult. First we prove the proposition that for any two loops γ, δ in X , we have $p_\gamma \cdot p_\delta \simeq p_{\gamma \cdot \delta}$. For consider a triangular region PQR in the xy -plane, one side of which, PR , is the unit interval in the x -axis. We define a map τ from PQR to X by

$$\tau(x, y) = \begin{cases} \gamma(2x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ \delta(2x - 1) & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

By restricting τ to the sides PQ , QR , and PR , we obtain the loops γ , δ , and $\gamma \cdot \delta$, respectively. So there is a 2-cell e_τ^2 in $L(X)$, homeomorphic to a triangle with sides attached to e_γ^1 , e_δ^1 , and $e_{\gamma \cdot \delta}^1$, respectively. This triangle affords a homotopy between a path tracing out the top two sides and a path tracing out the base. I.e., a homotopy $p_\gamma \cdot p_\delta \simeq p_{\gamma \cdot \delta}$.





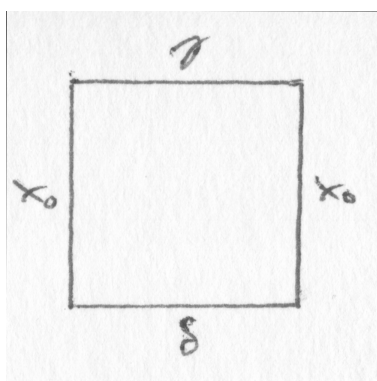
As a corollary, we have that $p_{x_0} \simeq *$, where x_0 denotes the constant path at x_0 , and $*$ denotes the constant path at the 0-cell of $L(X)$. To see this, note that

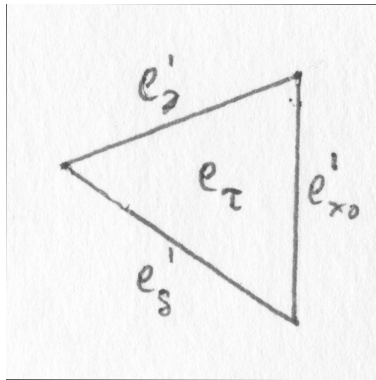
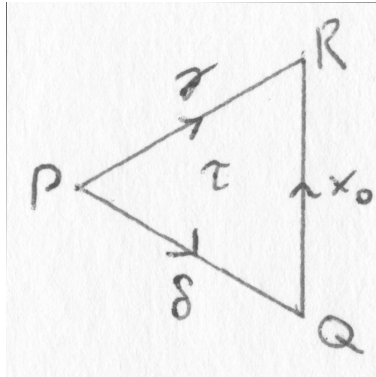
$$p_{x_0} \cdot p_{x_0} \simeq p_{x_0 \cdot x_0} = p_{x_0}.$$

Therefore

$$\begin{aligned} \overline{p_{x_0}} \cdot p_{x_0} \cdot p_{x_0} &\simeq \overline{p_{x_0}} \cdot p_{x_0} \\ p_{x_0} &\simeq *. \end{aligned}$$

Next we prove that if $\gamma \simeq \delta$, then $p_\gamma \simeq p_\delta$. The homotopy $\gamma \simeq \delta$ is a map from I^2 to X , which is constant on the side $\{0\} \times I$. So we can collapse that side and obtain a map τ from a filled triangle PQR to X , whose restrictions to PQ , QR , and PR are δ , x_0 , and γ , respectively. There is a corresponding cell e_τ^2 which affords a homotopy $p_\gamma \simeq p_\delta \cdot p_{x_0}$, and the latter path is homotopic to p_δ by the above paragraph.





As a corollary, we conclude that for any path γ , $p_\gamma \simeq \overline{p_\gamma}$. Indeed, we have

$$p_{\overline{\gamma}} \cdot p_\gamma \simeq p_{\overline{\gamma} \cdot \gamma} \simeq p_{x_0} \simeq *,$$

so

$$p_{\overline{\gamma}} \simeq \overline{p_\gamma}.$$

Now we are ready to prove injectivity of the map f_* . Suppose a is a loop in $L(X)$. Since $L(X)$ is a 2-dimensional CW-complex, the map induced by the inclusion of the 1-skeleton $\pi_1(L(X)^1) \rightarrow \pi_1(L(X))$ is surjective (quotient of the free group by the loops which attach the 2-cells). Therefore, a is homotopic to a loop in the 1-skeleton $L(X)^1$, which is a wedge of circles. The fundamental group of a wedge of circles is the free product on the circles, so a is homotopic to a concatenation $\varepsilon_{\gamma_1} \cdots \varepsilon_{\gamma_n}$, where ε_{γ_i} is either p_{γ_i} or $\overline{p_{\gamma_i}}$. And since $p_{\overline{\gamma_i}} \simeq \overline{p_{\gamma_i}}$, a is homotopic to a concatenation $p_{\delta_1} \cdots p_{\delta_n}$, where δ_i is either γ_i or $\overline{\gamma_i}$. And so $a \simeq p_\delta$, where $\delta = \delta_1 \cdots \delta_n$. Now if $f_*[a] = 1$, then $1 = f_*[a] = f_*[p_\delta] = [\delta]$. So $\delta \simeq x_0$, and so $a \simeq p_\delta \simeq p_{x_0} \simeq *$. Thus f is injective.

6. (10 marks) Show that the join $X * Y$ of two nonempty spaces X and Y is simply-connected if X is path-connected. Assume also that Y is locally path-connected.

Solution

We think of $X * Y$ as the product $X \times Y \times I / \sim$, and let p be the projection from $X \times Y \times I$ to $X * Y$. Let $A = X \times Y \times [0, 1] / \sim$ and $B = X \times Y \times (0, 1] / \sim$. Note that A and B are open and $X * Y = A \cup B$.

First we show that $X * Y$ is path connected. Let $x_0 \in X$ and $y_0 \in Y$, we will show that any point is path connected to $p(x_0, y_0, 0)$. Let $x \in X$, since X is path-connected there is a path α from x_0 to x . The path $p(\alpha(t), y_0, 0)$ connects $p(x_0, y_0, 0)$ to $p(x, y_0, 0) = p(x, y_1, 0)$ for any $y_1 \in Y$. Now, let $p(x, y, s) \in B$, that is, $s \neq 0$. Then the path $p(x, y, st)$ connects $p(x, y, s)$ and $p(x, y, 0)$. This proves the claim.

Now assume that Y is path-connected. Choose a basepoint $(x_0, y_0, \frac{1}{2})$. Then A deformation-retracts onto X , B deformation-retracts onto Y . $A \cap B = X \times Y \times (0, 1)$ deformation-retracts onto $X \times Y \times \{\frac{1}{2}\} \approx X \times Y$, which is path-connected because X and Y are. For example, take the homotopy $H : A \times I \rightarrow A$ defined by $H(p(x, y, t), s) = p(x, y, st)$. This is well defined, because $H(p(x, y, 0), s) = H(p(x, y_2, 0), s)$ for all s . Now, the image of H_0 is contained in $X = p(X \times Y \times \{0\})$, H_t restricted to X is the identity and H_1 is the identity. Similar homotopies can be defined for A and B .

So $\pi_1(A \cap B) \approx \pi_1(X) \times \pi_1(Y)$, $\pi_1(A) \approx \pi_1(X)$, and $\pi_1(B) \approx \pi_1(Y)$. Under these identifications, considering the deformation retracts mentioned above, the inclusion $A \cap B \hookrightarrow A$ induces the projection $\pi_1(X) \times \pi_1(Y) \rightarrow \pi_1(X)$; likewise, the inclusion $A \cap B \hookrightarrow B$ induces the projection $\pi_1(X) \times \pi_1(Y) \rightarrow \pi_1(Y)$. Van Kampen tells us that

$$\pi_1(X * Y) \approx \pi_1(X) * \pi_1(Y) / N$$

where N is generated by elements of the form ab^{-1} , where a is any element of $\pi_1(X)$ and b is any element of $\pi_1(Y)$. So $N = \pi_1(X) * \pi_1(Y)$ and $\pi_1(X * Y) = 1$.

Now consider the case where Y is the union of path components $\cup_{\tau} Y_{\tau}$. These are open if Y is locally path connected. Let $C_{\tau} = X * Y_{\tau}$. By the above paragraph, we know that $\pi_1(C_{\tau}) \approx \pi_1(X * Y_{\tau}) = 1$. The sets C_{τ} cover $X * Y$ and are path-connected. $C_{\tau} \cap C_{\tau'} = X$ if $\tau \neq \tau'$ and same for triple intersections. Van Kampen's theorem applies; and since all the C_{τ} are simply-connected, the theorem tells us that $\pi_1(X * Y)$ is simply-connected as well.