MATH 215B MIDTERM SOLUTIONS

1.

(a) (6 marks) Show that a finitely generated group has only a finite number of subgroups of a given finite index. (Hint: Do it for a free group first.)

(b) (6 marks) Show that an index n subgroup H of a group G has at most n conjugate subgroups gHg^{-1} in G. Apply this to show that there exists a normal subgroup $K \subset G$ of finite index in G with $K \subset H$.

Solution

(a) Suppose we have a finitely-generated free group $*_n\mathbb{Z}$, and a subgroup H of finite index k. We can build a finite graph X having fundamental group $*_n\mathbb{Z}$; it is a wedge of n circles, having 1 vertex and n edges. The subgroup H determines a connected based covering space $\widetilde{X} \xrightarrow{p} X$, which is also a graph. The number of sheets is the index of $p_*\pi_1(\widetilde{X})$ in $\pi_1(X)$, which is k. So $\widetilde{X}$ is a based graph with k vertices and nk edges. Up to based isomorphism, there are only finitely many based graphs with k vertices and nk edges. So by the classification of covering maps, there are only finitely many subgroups H of index k.

Now suppose we have a finitely-generated group. Every finitely-generated group is a quotient of a finitely-generated free group, so we can assume our group is $*_n\mathbb{Z}/K$ for some normal subgroup $K \subset *_n\mathbb{Z}$. An index k subgroup of $*_n\mathbb{Z}/K$ is of the form H/K where H is an index k subgroup of $*_n\mathbb{Z}$ that contains K. By the previous paragraph, there are only finitely many of these.

(b) Since any element in G has an inverse, it is equivalent to show that H has at most n conjugate subgroups $g^{-1}Hg$.

By corollary 1.28, there is a 2-dimensional CW-complex X with $\pi_1(X, x_0) \cong G$ for some $x_0 \in X$. Since it is a CW-complex, X is locally path-connected and semilocally simply-connected (because it is locally contractible). Note that the CW-complex constructed in this proof is also path-connected, so the correspondence theorem applies and so there is a path-connected n-sheeted cover $p : \tilde{X} \to X$ with $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

Let $g \in G \cong \pi_1(X, x_0)$. This element is represented by the homotopy class of a loop α at x_0. By the path-lifting property for covering spaces, there is a lift $\tilde{\alpha} : I \to \tilde{X}$ with $\tilde{\alpha}(0) = \tilde{x}_0$. Let $\tilde{x}_1 = \tilde{\alpha}(1)$. Note that $p(\tilde{x}_1) = x_0$.

$$g^{-1}Hg = [\alpha]^{-1}p_*(\pi_1(\tilde{X}, \tilde{x}_0))[\alpha] = p_*([\tilde{\alpha}]^{-1}\pi_1(\tilde{X}, \tilde{x}_0)[\tilde{\alpha}]) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$$

So for any $g \in G$, $g^{-1}Hg = p_*(\pi_1(\tilde{X}, y))$ for some $y \in \tilde{X}$ with $p(y) = x_0$. Since p is an n-sheeted cover, there are only n such y's, and so there are at most n subgroups $g^{-1}Hg$ in G.

Now consider $K = \bigcap_{g \in G} g^{-1}Hg$. This subgroup of G is normal by construction and $K \leq H$. By the previous argument, we actually know that $K = \bigcap_{i=1}^n g_i^{-1} H g_i$ for some $g_i \in G$. Let $K_i = g_i^{-1} H g_i$. Now consider the map of **sets** $\Phi : G/K \to \prod_{i=1}^n G/K_i$ that takes gK to the n-tuple $(gK_1, gK_2, \ldots, gK_n)$.

Φ is well defined. If $gK = xK$, then $x = gk$ for some $k \in K = \bigcap_{i=1}^n K_i$ and so $\Phi(xK) = (gkK_1, gkK_2, \ldots, gkK_n) = (gK_1, gK_2, \ldots, gK_n) = \Phi(gK)$.

Φ is injective. If $\Phi(gK) = \Phi(mK)$, then $gm^{-1} \in K_i$ for all i and so $gm^{-1} \in K \Rightarrow gK = mK$.

Therefore $[G : K] \leq \prod_{i=1}^n [G : K_i] = n^n$ and so K has finite index in G.

2. (10 marks) Show that a local homeomorphism $f : X \to Y$ between compact Hausdorff spaces is a covering space. (Note: A local homeomorphism is a map such that for each $x \in X$, there are open neighborhoods U of x in X and V of $f(x)$ in Y with $f : U \to V$ a homeomorphism.)

Solution

Suppose $y \in Y$. Y is Hausdorff, so $\{y\}$ is closed in Y. f is continuous, so $f^{-1}(y)$ is closed in X. X is compact, so $f^{-1}(y)$ is compact.

For each $x \in f^{-1}(y)$, there is a neighborhood U_x of x and a neighborhood V_x of y such that $f\big|_{U_x} : U_x \xrightarrow{\approx} V_x$ is a homeomorphism. Any restriction of $f\big|_{U_x}$ is also a homeomorphism onto its image.

$f^{-1}(y)$ is compact, so is covered by finitely many opens $U_{x_1}, \ldots, U_{x_n}$. Each U_{x_i} meets f^{-1} in exactly one point, namely x_i. So $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_n\}$. The U_{x_i} are open sets that are mapped homeomorphically by f onto their images, but they might not be disjoint.

X is Hausdorff, so we can cover $f^{-1}(y)$ with pairwise disjoint opens A_i such that $A_i \cap f^{-1}(y) = \{x_i\}$. Let $U_i' = U_{x_i} \cap A_i$. Then the U_i' are mutually disjoint open sets that are mapped homeomorphically by f onto their images, but they might not cover the preimage of a neighborhood of y.

Let $K = X - \cup U_i'$. It is closed and thus compact, and so $f(K)$ is compact and thus a closed subset of Y. And so $V = \cap f(U_i') - f(K)$ is an open subset of Y. Also, $f^{-1}(y) \subset \cup U_i'$, so $f^{-1}(y) \cap K = \emptyset$, so $y \notin f(K)$, so $y \in V$. Furthermore, $f^{-1}(V) \cap K = \emptyset$, so the U_i' cover $f^{-1}(V)$.

Thus V is a neighborhood of y and $f^{-1}(V)$ is a disjoint union of open sets $U_i' \cap f^{-1}(V)$ which are mapped homeomorphically by f onto V. Thus f is a covering map.

3. (8 marks) Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence 1, $1/2$, $1/3$, $\cdots$ together with its limit 0. (Hint: See example 1.25).

Solution

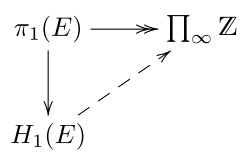
From the long exact sequence of the pair (X, A),

$$0 \to H_1(X, A) \to H_0(A) \to H_0(X)$$

we can see that $H_1(X, A)$ is isomorphic to the kernel of $H_0(A) \to H_0(X)$. Now $H_0(A) \approx \oplus_\infty \mathbb{Z}$ is the free abelian group generated by the components of A; and $H_0(X) \approx \mathbb{Z}$ is generated by the single component of X; and the map between them sends each generator of $\oplus_\infty \mathbb{Z}$ to the generator of $\mathbb{Z}$. So the kernel (and therefore $H_1(X, A)$) is a free abelian group on a countable infinity of generators. If we give names $a_1, a_2, \dots$ to the generators of $H_0(A)$, then the kernel is generated by the elements $a_1 - a_n$ for all n. In particular, it is countable.

On the other hand, X/A is homeomorphic to the Hawaiian earring E of Example 1.25. (Note that in X, every neighborhood of the origin contains some infinite sequence $\frac{1}{N}, \frac{1}{N+1}, \dots$ of points of A. Correspondingly in E, every neighborhood of the origin contains an infinite descending sequence of circles.) It was proved in Example 1.25 that $\pi_1(E)$ surjects onto the uncountable group $\prod_\infty \mathbb{Z}$.

By Section 2.A, $H_1(E)$ is isomorphic to the abelianization of $\pi_1(E)$. Recall the universal property of abelianization: If G is a group and B is an abelian group, then every homomorphism $G \to B$ factors through the abelianization: $G \to G_{\text{ab}} \to B$. Thus our surjection factors through homology:



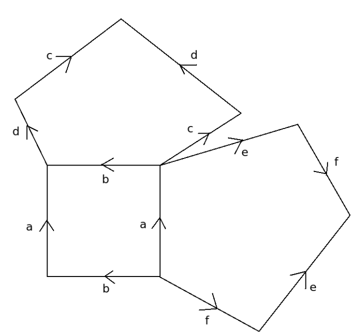
Since the top map is surjective, the diagonal map is also surjective. Therefore $\widetilde{H}_1(X/A) \approx \widetilde{H}_1(E) = H_1(E)$ surjects onto the uncountable group $\prod_\infty \mathbb{Z}$, and is itself uncountable.

Since $H_1(X, A)$ and $H_1(X/A)$ are different, (X, A) is not a good pair.

4. (8 marks) Let T be the torus $S^1 \times S^1$ and let T' be T with a small open disk removed. Let X be obtained from T by attaching two copies of T', identifying their boundary circles with longitude and meridian circles $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$ in T. Compute $\pi_1(X)$.

Solution

Let a be the loop going around the longitude circle $S^1 \times \{x_0\}$ and b the loop going around the meridian circle $\{x_0\} \times S^1$ in T. We identify a with the boundary circle of a copy of T', and b with the boundary of the other copy of T'. The result is the following identification space:



This is CW-complex with one 0-cell P, six 1-cells a, b, c, d, e and f and three 2-cells attached via the loops $aba^{-1}b^{-1}$, $aefe^{-1}f^{-1}$ and $bdcd^{-1}c^{-1}$. By proposition 1A.2, the fundamental group of the 1-skeleton is free on generators a, b, c, d, e and f. By proposition 1.26:

$\pi_1(X) \cong < a, b, c, d, e, f \mid aba^{-1}b^{-1}\ ,\ aefe^{-1}f^{-1}\ ,\ bdcd^{-1}c^{-1} > =$

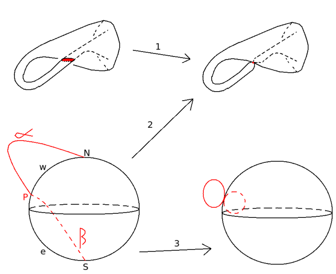
$= < c, d, e, f \mid \big[[f, e], [c, d]\big] >$

5. Let $X \subset \mathbb{R}^3$ be the subspace given by the Klein bottle intersecting itself in a circle in $\mathbb{R}^3$. (See the picture in exercise 20, page 19)

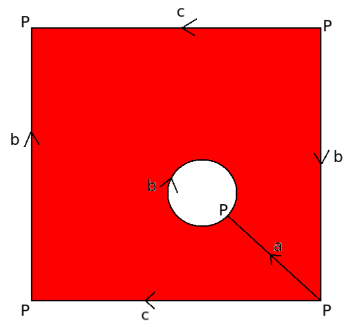
(a) (4 marks) Show that X is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.
(b) (8 marks) Remove the disk in the Klein bottle bounded by the circle of self-intersection in X. Denote the resulting space by Y. Show that $\pi_1(Y)$ has a presentation given by $< a, b, c \mid aba^{-1}b^{-1}cb^{\epsilon}c^{-1} >$, where $\epsilon = \pm 1$ (Either choice of sign is ok).

Solution

(a) We follow this process, shown in the picture at the end.
Step 1: Note that X is a CW-complex and the disk bound by the self-intersection circle can be considered as a subcomplex (see picture in part b). Since the disk is contractible, by propositions 0.16 and 0.17, we can contract it to a point. Call the resulting space $\tilde{X}$.
Step 2: Now consider S^2 with an arc α attached from the north pole N to some point P which is not N or the south pole S, and another arc β attached from S to P inside the sphere. This space Z is a CW-complex. Consider the great circle containing these three points and let w be the part that goes P to N and e from S to P.
The subcomplex A consisting of $\{N, S, P\}$ and the arcs α and β is contractible., then the projection $Z \to Z/A$ is a homotopy equivalence, and Z/A is clearly homotopy equivalent to $\tilde{X}$.
Step 3: The subspace B consisting of $\{N, S, P\}$ and the arcs w and e is contractible. So $Z \simeq Z/B$. But note that Z/B is homotopic to $S^2 \vee S^1 \vee S^1$.
Therefore $X \simeq \tilde{X} \simeq Z/A \simeq Z \simeq Z/B \simeq S^2 \vee S^1 \vee S^1$.



(b) This space is homeomorphic to the following space with identifications, where the red part is including in the space, but the white is not:



But this is a CW-complex with one 0-cell P, three 1-cells a, b, c attached to P and one 2-cell attached to the loop $aba^{-1}b^{-1}cb^{-1}c^{-1}$. By propositions 1.A.2 and 1.26:

$\pi_1(Y) =< a, b, c \mid aba^{-1}b^{-1}cb^{-1}c^{-1} >$

6. (12 marks) Describe all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ up to isomorphism of covering spaces.

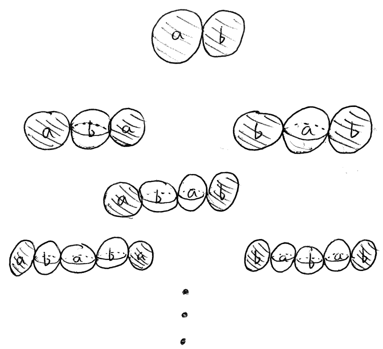
Solution

$\pi_1(\mathbb{R}P^2)$ is isomorphic to $\mathbb{Z}/2$, which has two subgroups. So there are two connected covering spaces of $\mathbb{R}P^2$, up to isomorphism. They are (1) the identity $\mathbb{R}P^2 \to \mathbb{R}P^2$ and (2) the two-sheeted cover $S^2 \to \mathbb{R}P^2$ which is the quotient by the antipodal action. In covering space (1) the preimage of the basepoint is the basepoint itself, and in covering space (2) the preimage of the basepoint is the two poles of the sphere. Any covering space of $\mathbb{R}P^2$ is a disjoint union of copies of these spaces.

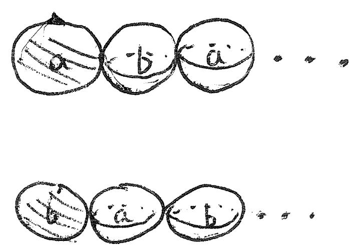
Let us use the labels a and b to distinguish between the two wedge summands of $\mathbb{R}P^2 \vee \mathbb{R}P^2$. A covering space of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ restricts to a covering space of each summand. In other words, any such covering space is the union of a covering space of a and a covering space of b. So it's the union of a bunch of copies of $\mathbb{R}P^2$ and S^2, which we may label a and b to indicate how the covering map goes.

a and b intersect at the basepoint. The preimage of a neighborhood U of the basepoint in a covering space must be a disjoint union of neighborhoods that are homeomorphic to U. So the copies of $\mathbb{R}P^2$ and S^2 comprising the covering space can intersect in only a few ways: The two basepoints of two copies of $\mathbb{R}P^2$ may be identified, or one pole from each of two copies of a sphere may be identified, or the basepoint of a copy of $\mathbb{R}P^2$ can be identified with a pole of a sphere. In every case the two spaces involved in an intersection must bear different labels.

Since S^2 has two poles and $\mathbb{R}P^2$ has only one basepoint, it is easy to list all the possible ways of putting them together. We can have a chain of spheres, terminated on either end with a copy of $\mathbb{R}P^2$:

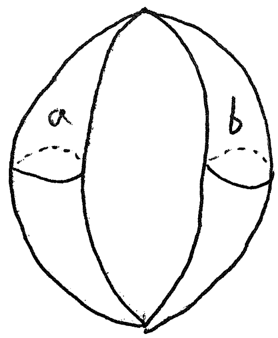


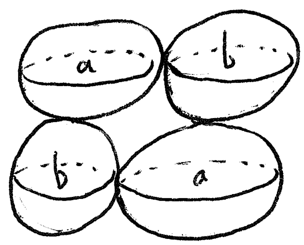
(If there is an odd number of spheres, there are two nonisomorphic ways of labelling the components.) Or we can have an infinite chain, terminated at one end with a copy of $\mathbb{R}P^2$:

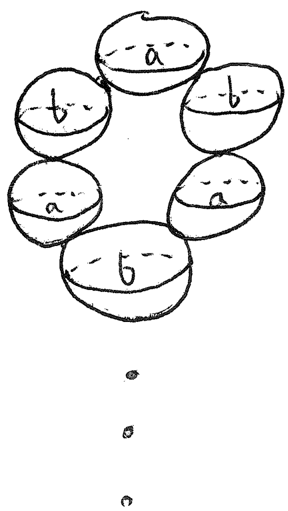


Or we can have a chain of spheres that does not terminate:

(This is the universal cover.) Finally, we can have a closed loop consisting of an even number of spheres:







(For some of these covering spaces, there are multiple nonequivalent choices of basepoint.)