

## TAKE-HOME MIDTERM DUE FEBRUARY 16TH BY 5PM

Please remember to write down your name and your Stanford ID number (9 digits). You may use any results in the book included in chapters 0, 1, 2.1 and 2.A, but you have to work on your own.

1.
  - (a) (6 marks) Show that a finitely generated group has only a finite number of subgroups of a given finite index. (Hint: Do it for a free group first.)
  - (b) (6 marks) Show that an index  $n$  subgroup  $H$  of a group  $G$  has at most  $n$  conjugate subgroups  $gHg^{-1}$  in  $G$ . Apply this to show that there exists a normal subgroup  $K \subset G$  of finite index in  $G$  with  $K \subset H$ .
2. (10 marks) Show that a local homeomorphism  $f : X \rightarrow Y$  between compact Hausdorff spaces is a covering space. (Note: A local homeomorphism is a map such that for each  $x \in X$ , there are open neighbourhoods  $U$  of  $x$  in  $X$  and  $V$  of  $f(x)$  in  $Y$  with  $f : U \rightarrow V$  a homeomorphism)
3. (8 marks) Show that  $H_1(X, A)$  is not isomorphic to  $\tilde{H}_1(X/A)$  if  $X = [0, 1]$  and  $A$  is the sequence  $1, 1/2, 1/3, \dots$  together with its limit 0. (Hint: See example 1.25).
4. (8 marks) Let  $T$  be the torus  $S^1 \times S^1$  and let  $T'$  be  $T$  with a small open disk removed. Let  $X$  be obtained from  $T$  by attaching two copies of  $T'$ , identifying their boundary circles with longitude and meridian circles  $S^1 \times \{x_0\}$  and  $\{x_0\} \times S^1$  in  $T$ . Compute  $\pi_1(X)$ .
5. Let  $X \subset \mathbb{R}^3$  be the subspace given by the Klein bottle intersecting itself in a circle in  $\mathbb{R}^3$ . (See the picture in exercise 20, page 19)
  - (a) (4 marks) Show that  $X$  is homotopy equivalent to  $S^1 \vee S^1 \vee S^2$ .

- (b) (8 marks) Remove the disk in the Klein bottle bounded by the circle of self-intersection in  $X$ . Denote the resulting space by  $Y$ . Show that  $\pi_1(Y)$  has a presentation given by  $\langle a, b, c \mid aba^{-1}b^{-1}cb^\epsilon c^{-1} \rangle$ , where  $\epsilon = \pm 1$  (Either choice of sign is ok).
6. (12 marks) Describe all the connected covering spaces of  $\mathbb{R}P^2 \vee \mathbb{R}P^2$  up to isomorphism of covering spaces.