HOMEWORK 3 DUE FEBRUARY 7TH BY 5PM

Please remember to write down your name and your Stanford ID number (9 digits). All pages refer to Hatcher's book. You may use any results in the book included in chapters 0 and 1.

1. (6 marks) Exercise 9, page 79.
2. (10 marks) Exercise 12, page 80.
3. (12 marks) Let $U_+ = \{z \in S^1 \mid Im(z) > 0\}$ and $U_- = \{z \in S^1 \mid Im(z) < 0\}$. Let $\sim$ be the equivalence relation on S^1 where $x \sim y$ if and only if either ($x \in U_+$ and $y \in U_+$) or ($x \in U_-$ and $y \in U_-$) or $x = y$. Let $X = S^1/\sim = \{[1], [-1], U_+, U_-\}$ have the quotient topology. Let $q : S^1 \to X$ be the quotient map. Prove that $q_* : \pi_1(S^1, 1) \to \pi_1(X, [1])$ is an isomorphism.
4. (12 marks) Exercise 18, page 80. Ignore the $S^1 \vee S^1 \vee S^1$ part, but do it for $S^1 \vee S^1$.
5. A section of a covering space $p : E \to X$ is a continuous map $s : X \to E$ such that $ps(x) = x$ for all $x \in X$. Let $p : E \to X$ be a covering space over a path connected, locally path connected and semi-locally simply connected X and let $x_0 \in X$.

(a) (6 marks) Show that there is a bijection between the set of sections of p and the set of $y \in p^{-1}(x_0)$ such that any lift of a loop based at x_0 that starts at y must be a loop.

(b) (4 marks) Given a positive integer $n \geq 2$, deduce that there are no continuous maps $f : S^1 \to S^1$ such that $f(x)^n = x$ for all x, where $f(x)^n$ is the nth power of $f(x)$ as a complex number.

6. For a finite graph X define the Euler characteristic $\chi(X)$ to be the number of vertices minus the number of edges.

(a) (7 marks) Show that $\chi(X) = 1$ if X is a tree, and that the rank of $\pi_1(X)$ (the number of elements in a basis) is $1 - \chi(X)$ if X is path connected. Deduce that the Euler characteristic is a homotopy invariant of path connected graphs.

(b) (3 marks) Let n and k be positive integers, and let F be the free group on n generators. Show that if G is an index k subgroup of F, then G is a free group on $kn - k + 1$ generators.