WINTER 2012 MATH 215B FINAL EXAM SOLUTIONS

1. (12 marks) Let X and Y be path-connected and locally contractible spaces such that $H^1(X, \mathbb{Q}) \neq 0$ and $H^1(Y, \mathbb{Q}) \neq 0$. Show that $X \vee Y$ is not a retract of $X \times Y$.

Solution

First we show that $\pi_1(X)$ and $\pi_1(Y)$ have subgroups isomorphic to $\mathbb{Z}$.

By the Universal coefficient theorem for cohomology, since $H_0(X) \cong \mathbb{Z}$ is free abelian, $H^1(X, \mathbb{Q}) \cong \mathrm{Hom}(H_1(X), \mathbb{Q})$. Since $\mathbb{Q}$ is abelian and $H_1(X)$ is the abelianization of $\pi_1(X)$, we also have $H^1(X, \mathbb{Q}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Q})$.

Given any homomorphism $f : \pi_1(X) \to \mathbb{Q}$, and any element $a \in \pi_1(X)$ with finite order, we must have $f(a) = 0$ since $\mathbb{Q}$ is a torsion-free abelian group. Since $H^1(X; Q) \neq 0$, this implies that there must be at least one element of infinite order in $\pi_1(X)$. Let $\phi : \mathbb{Z} \to \pi_1(X)$ the only homomorphism that takes 1 to that element. This is a monomorphism since the element has infinite order, and the image of ϕ is therefore isomorphic to $\mathbb{Z}$. The same argument applies to Y.

Let $i : X \vee Y \to X \times Y$ be the standard inclusion and assume there existed $r : X \times Y \to X \vee Y$ such that $ri = 1$.

We showed that $\pi_1(X)$ and $\pi_1(Y)$ both have subgroups isomorphic to $\mathbb{Z}$. By van Kampen's theorem, $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$ and we can deduce then that $\pi_1(X \vee Y)$ has a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$. Since $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$, the image of this subgroup under i_* is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, which is commutative.

Let a, b be generators of each of these two copies of $\mathbb{Z}$. Then $r_* i_*(aba^{-1}b^{-1}) = r_*(0) = 0$, which is a contradiction with $ri = 1$ because $aba^{-1}b^{-1} \neq 0$ in $\pi_1(X \vee Y)$.

2. (12 marks) The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a space X. Compute the homology groups of X and the relative homology groups of (R, M_g).

Solution

We take A and B to be open neighborhoods in X that deformation-retract onto the copies of R, and whose intersection deformation-retracts onto $M_g \subset X$. R deformation-retracts onto a wedge of g copies of S^1. Therefore our Mayer-Vietoris sequence looks like

$$\cdots \to H_*(M_g) \to H_*(\vee_g S^1) \oplus H_*(\vee_g S^1) \to H_*(X) \to \cdots$$

Now one bit of the Mayer-Vietoris sequence looks like

$$0 \to H_3(X) \to H_2(M_g) \to 0$$

so $H_3(X) \cong H_2(M_g) \cong \mathbb{Z}$. Another bit of the (reduced) Mayer-Vietoris sequence looks like

$$0 \to H_2(X) \to \mathbb{Z}^{2g} \to \mathbb{Z}^g \oplus \mathbb{Z}^g \to H_1(X) \to 0$$

We need to know what the middle map is. $H_1(M_g) \cong \mathbb{Z}^{2g}$ is generated by two kinds of loops: There are g "latitudinal" generators and g "longitudinal" generators. (In the figure on page 5 of Hatcher, c is a latitudinal loop and d is a longitudinal loop.) Call the latitudinal generators $a_1, \ldots, a_g$ and call the longitudinal generators $b_1, \ldots, b_g$. Now $H_1(R) \cong \mathbb{Z}^g$ is generated by only the latitudinal loops. So the inclusion $M_g \hookrightarrow R$ sends each a_i to a different generator of H_1, and sends each b_i to 0.

Thus the middle map is of rank g. $H_2(X)$ is isomorphic to the kernel, $\mathbb{Z}^g$, and $H_1(X)$ is isomorphic to the cokernel, $\mathbb{Z}^g$.

X is connected, so $H_0(X) \cong \mathbb{Z}$. And from the Mayer-Vietoris sequence it can be seen that the homology in dimensions higher than 3 is trivial. (This is also because X is a 3-manifold.) In summary:

$$H_i(X) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^g & \text{if } i = 1, 2 \\ \mathbb{Z} & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

As for $H_i(R, M_g)$, we have the reduced long exact sequence of the pair, one bit of which looks like

$$0 \to H_3(R, M_g) \to \mathbb{Z} \to 0$$

so $H_3(R, M_g) \cong \mathbb{Z}$. Another bit looks like

$$0 \to H_2(R, M_g) \to H_1(M_g) \to H_1(R) \to H_1(R, M_g) \to 0$$

As we saw before, the middle map is a rank g map from $\mathbb{Z}^{2g}$ to $\mathbb{Z}^g$. So $H_2(R, M_g)$ is isomorphic to the kernel $\mathbb{Z}^g$, and $H_1(R, M_g)$ is isomorphic to the cokernel, which is 0.

The group $H_0(R, M_g)$ is trivial because R and M_g each have one path-component, and the homology above dimension 3 vanishes.

$$H_i(R, M_g) \cong \begin{cases} \mathbb{Z}^g & \text{if } i = 2 \\ \mathbb{Z} & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

3. (8 marks) Show that the spaces $S^2 \times \mathbb{R}P^4$ and $S^4 \times \mathbb{R}P^2$ are not homotopy equivalent.
Solution

We use cohomology with coefficients in $\mathbb{Z}/2$. The cohomology groups of S^n are all free and finitely generated, so we can use Kunneth theorem:

$$H^*(S^2 \times \mathbb{R}P^4; \mathbb{Z}/2) \cong H^*(S^2; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^*(\mathbb{R}P^4; \mathbb{Z}/2) \cong \Lambda_{\mathbb{Z}/2}(a_2) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[b_1]/(b_1^5)$$

$$H^*(S^4 \times \mathbb{R}P^2; \mathbb{Z}/2) \cong H^*(S^4; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \Lambda_{\mathbb{Z}/2}(c_4) \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[d_1]/(d_1^3)$$

where the degree of the generators is indicated by their subindex. If there was a homotopy equivalence $f : S^2 \times \mathbb{R}P^4 \to S^4 \times \mathbb{R}P^2$, then f^* would be a graded ring isomorphism, so in particular $f^*(d_1)$ should be a generator of $H^1(S^2 \times \mathbb{R}P^4; \mathbb{Z}/2)$, that is $f^*(d_1) = b_1$.

But now $0 = f^*(0) = f^*(d_1^3) = b_1^3 \neq 0$, which is a contradiction. This proves that $S^2 \times \mathbb{R}P^4$ and $S^4 \times \mathbb{R}P^2$ are not homotopy equivalent.

4. (12 marks) Let $f : \mathbb{R}P^{2n} \to Y$ be a covering map, where Y is a path-connected CW-complex, $n > 0$. Prove that f must be a homeomorphism. (Careful: When is the composition of covering maps a covering map?)

Solution

First we show that if $g : X \to Z$ is a covering map and $h : Z \to B$ is a finite-sheeted covering map, the composition $hg : X \to B$ is a covering map. Let $b \in B$ and choose a distinguished neighbourhood U with respect to h. That is, $h^{-1}(U) = V_1 \amalg \ldots \amalg V_k$ and $h : V_i \to U$ is a homeomorphism. Let $h^{-1}(x) = \{y_1, \ldots, y_k\}$ where $y_i \in V_i$. Pick distinguished neighbourhoods W_i of y_i with respect to g with $W_i \subseteq V_i$, say $g^{-1}(W_i) = \coprod_j A_{ij}$ and $g : A_{ij} \to W_i$ is a homeomorphism. Note that h restricted to W_i is a homeomorphism onto its image and $h(W_i)$ is open in B. Let $U' = h(W_1) \cap \ldots \cap h(W_n)$. Then U' is an distinguished neighbourhood of x with respect to h and actually with respect to hg too. Let $W_i' = W_i \cap h^{-1}(U')$ and $A_{ij}' = g^{-1}(W_i')$. Then $g^{-1}h^{-1}(U') = g^{-1}(W_1' \amalg \ldots \amalg W_n') = \coprod_{i=1}^n A_{ij}'$. And $hg : A_{ij}' \to W_i' \to U'$ are homeomorphisms.

Now we show that f must be a finite-sheeted covering map. For each element in $\mathbb{R}P^{2n}$ choose an open neighbourhood such that f is a homeomorphism restricted to that neighbourhood. By the compactness of $\mathbb{R}P^{2n}$, it is a covered by a finite number of such neighbourhoods and therefore f has a finite number of sheets.

Consider the universal cover $\pi : S^{2n} \to \mathbb{R}P^{2n}$. Since f is finite-sheeted, the composition $p = f \circ \pi : S^{2n} \to Y$ is a covering space, which must be the universal covering space of Y. Since Y is a CW-complex, it is locally path-connected and we can use proposition 1.39 to conclude that p is a normal covering space and the group G of covering transformations of p is isomorphic to $\pi_1(Y)$.

Since the action of G on S^{2n} is a covering action, it is a free action. Since this cover factors through $\mathbb{R}P^{2n}$, $G \neq \{1\}$. By proposition 2.29, $G \cong \mathbb{Z}/2$ and so p is a 2-sheeted cover, which implies that f is a 1-sheeted cover, that is, f is a homeomorphism.

5. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z^n denotes the nth power of a complex number and we consider the standard inclusion $S^1 \subseteq \mathbb{C}$.

(a) (6 marks) Compute the fundamental group and the cohomology ring of X with coefficients in $\mathbb{Z}$.

Solution

X is a path-connected CW-complex with one 1-cell, so $\pi_1(X^1) \cong \mathbb{Z}$. We are attaching two cells whose attaching maps generate the subgroups $3\mathbb{Z}$ and $5\mathbb{Z}$, respectively. These two subgroups together generate $\mathbb{Z}$, so by proposition 1.26, $\pi_1(X) = 0$.

Since X is path-connected, $H^0(X;\mathbb{Z}) \cong \mathbb{Z}$. By the universal coefficient theorem, $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X), \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}) = 0$. Since $H_1(X) = 0$, we also have $H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X), \mathbb{Z})$.

We compute $H_2(X)$ using cellular homology. We have $C_2^{CW}(X) = \mathbb{Z} < e_1 > \oplus \mathbb{Z} < e_2 >$ and $C_1^{CW}(X) = \mathbb{Z} < e >$ and $d_2(e_1) = 3e$, $d_2(e_2) = 5e$. The kernel of d_2 is the subgroup generated by $(5e_1, -3e_2)$, so $H_2(X) \cong \mathbb{Z}$, hence $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$.

Since X is a 2-dimensional CW-complex, $H^n(X;\mathbb{Z}) = 0$ for $n > 2$. Because we only have cohomology in degrees 0 and 2, the only nontrivial products are those with classes of degree 0 which correspond to multiplication by elements of $\mathbb{Z}$. Therefore

$$H^*(X;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(a_2)$$

where a_2 is a generator of $H^2(X;\mathbb{Z})$.

(b) (6 marks) Show that X is not homeomorphic to S^2.

Solution

If X was homeomorphic to S^2, let p be a point in the interior of the 2-cell attached via the map $z \mapsto z^3$. Then $X - \{p\}$ would be homeomorphic to $S^2 - \{q\}$, which is contractible. But $X - \{p\}$ is homotopy equivalent to the CW-complex obtained by attaching a 2-cell to S^1 via the map $z \mapsto z^5$. Therefore $\pi_1(X - \{p\}) \cong \mathbb{Z}/5$ and so it is not contractible.