HOMOTOPIC MAPS INDUCE CHAIN-HOMOTOPIC MAPS

Lemma 1. *If f, $g : X \to Y$ are homotopic, then $f_\sharp$ and $g_\sharp$ are chain-homotopic.*

Proof. First we break $\Delta^n \times I = [v_0, \ldots, v_n] \times I$ into $(n+1)$-simplices.

Let $v_i = (v_i, 0)$ and $w_i = (v_i, 1)$. Then we claim that $\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \ldots, v_i, w_i, \ldots, w_n]$.

Any element of $\Delta^n \times I$ is of the form $\left(\sum_{i=0}^{n} t_i v_i, t \right)$ with $\sum_{i=0}^{n} t_i = 1$, all $t_i \geq 0$ and $0 \leq t \leq 1$.

Since $0 \leq t \leq \sum_{i=0}^{n} t_i$, there must be some j such that $\sum_{i=j+1}^{n} t_i \leq t \leq \sum_{i=j}^{n} t_i$.

Then consider the following element of $[v_0, \ldots, v_j, w_j, \ldots, w_n]$

$$\sum_{i=0}^{j-1} t_i v_i + \left(\sum_{i=j}^{n} t_i - t \right) v_j + \left(t - \sum_{i=j+1}^{n} t_i \right) w_j + \sum_{i=j+1}^{n} t_i w_i =$$

$$\left(\sum_{i=0}^{j-1} t_i v_i + \left[\sum_{i=j}^{n} t_i - t \right] v_j + \left[t - \sum_{i=j+1}^{n} t_i \right] v_j + \sum_{i=j+1}^{n} t_i v_i, \left[t - \sum_{i=j+1}^{n} t_i \right] + \sum_{i=j+1}^{n} t_i \right) =$$

$$\left(\sum_{i=0}^{n} t_i v_i, t \right)$$

So $\left(\sum_{i=0}^{n} t_i v_i, t \right) \in [v_0, \ldots, v_j, w_j, \ldots, w_n]$ and this proves $\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \ldots, v_i, w_i, \ldots, w_n]$

Let $H : X \times I \to Y$ be such that $H_0 = f$ and $H_1 = g$. Given $\sigma : \Delta^n \to X$, let us abbreviate $H^\sigma = H \circ (\sigma \times 1_I)$. Define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i H^{\sigma}_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}$$

and extend it by linearity to a map $P : C_n(X) \to C_{n+1}(Y)$.

$$\partial P(\sigma) = \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{i} (-1)^j H^{\sigma}_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} + \sum_{i=0}^{n} (-1)^i \sum_{j=i+1}^{n+1} (-1)^j H^{\sigma}_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_{j-1}},\ldots,w_n]} =$$

We separate the first sum into two parts, when $j = i$ and when $j < i$. We separate the second sum into two parts, when $j = i + 1$ and when $j > i + 1$.

$$\sum_{i=0}^{n} (-1)^{i+i} H^{\sigma}_{[v_0,\ldots,\widehat{v_i},w_i,\ldots,w_n]} + \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} H^{\sigma}_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} + \sum_{i=0}^{n} (-1)^{i+i+1} H^{\sigma}_{[v_0,\ldots,v_i,\widehat{w_i},\ldots,w_n]} +$$
$$\sum_{i=0}^{n-1} \sum_{j=i+2}^{n+1} (-1)^{i+j} H^{\sigma}_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_{j-1}},\ldots,w_n]} =$$

Now note that the terms with $i > 0$ in the first summation $H^{\sigma}_{[v_0,\ldots,\widehat{v_i},w_i,\ldots,w_n]}$ are the same except for the sign as the $(i-1)$th term of the third summation $-H^{\sigma}_{[v_0,\ldots,v_{i-1},\widehat{w_{i-1}},\ldots,w_n]}$. Also note that when $i = 0$ in the first summation, we get $H^{\sigma}_{[\widehat{v_0},w_0,\ldots,w_n]} = g_\sharp(\sigma)$ and when $i = n$ in the third summation, we get $-H^{\sigma}_{[v_0,\ldots,v_n,\widehat{w_n}]} = -f_\sharp(\sigma)$.

$$g_\sharp(\sigma) + \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} H^{\sigma}_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} - f_\sharp(\sigma) + \sum_{i=0}^{n-1} \sum_{j=i+2}^{n+1} (-1)^{i+j} H^{\sigma}_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_{j-1}},\ldots,w_n]} =$$

Make a change of variable $k = j - 1$ in the fourth summand:

$$g_\sharp(\sigma) + \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} H^{\sigma}_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} - f_\sharp(\sigma) + \sum_{i=0}^{n-1} \sum_{k=i+1}^{n} (-1)^{i+j} H^{\sigma}_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_k},\ldots,w_n]}$$

We now compute $P(\partial\sigma)$. Recall $\partial\sigma = \sum_{j=0}^{n} (-1)^j \sigma_{[v_0,\ldots,\widehat{v_j},\ldots,v_n]}$. Note that:

$$H \circ (\sigma_{[v_0,\ldots,\widehat{v_j},\ldots,v_n]} \times 1_I)_{[v_0,\ldots,v_i,w_i,\ldots w_n]} = \begin{cases} H^\sigma_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_j},\ldots,w_n]} & \text{if } i < j \\ H^\sigma_{[v_0,\ldots,\widehat{v_j},\ldots,v_{i+1},w_{i+1},\ldots,w_n]} & \text{if } i \geq j. \end{cases}$$

Therefore:

$$P(\partial\sigma) = \sum_{j=1}^{n}(-1)^j \sum_{i=0}^{j-1}(-1)^i H^\sigma_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_j},\ldots,w_n]} + \sum_{j=0}^{n}(-1)^j \sum_{i=j}^{n-1}(-1)^i H^\sigma_{[v_0,\ldots,\widehat{v_j},\ldots,v_{i+1},w_{i+1},\ldots,w_n]} =$$

In the first summation, make $k = j$ and interchange the order of summation. Since $i < j = k$, now k ranges between $i+1$ and n. In the second summation, make $k = i+1$ and interchange the order of summation. Since $j \leq i$, now $j < k$ and j ranges between 0 and $k-1$.

$$\sum_{i=0}^{n-1} \sum_{k=i+1}^{n} (-1)^{i+k} H^\sigma_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_k},\ldots,w_n]} + \sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^{j+k-1} H^\sigma_{[v_0,\ldots,\widehat{v_j},\ldots,v_k,w_k,\ldots,w_n]}$$

Now note that the first summation is identical to the fourth summand of $\partial P(\sigma)$ with opposite sign, and the second summation is identical to the second summand of $\partial P(\sigma)$ with opposite sign. Therefore:

$$\partial P(\sigma) + P(\partial\sigma) = g_\sharp(\sigma) - f_\sharp(\sigma)$$

That is, $\partial P + P\partial = g_\sharp - f_\sharp$, which proves the lemma. $\square$