Overview

- Mathematical and computational theory, and applications to combinatorial, non-convex and nonlinear problems
  - Semidefinite programming
  - Real algebraic geometry
  - Duality and certificates

Topics

1. Convexity and duality
2. Quadratically constrained quadratic programming
3. From duality to algebra
4. Algebra and geometry
5. Sums of squares and semidefinite programming
6. Polynomials and duality; the Positivstellensatz

Discrete Problems: LQR with Binary Inputs

- linear discrete-time system $x(t+1) = Ax(t) + Bu(t)$ on interval $t = 0, \ldots, N$

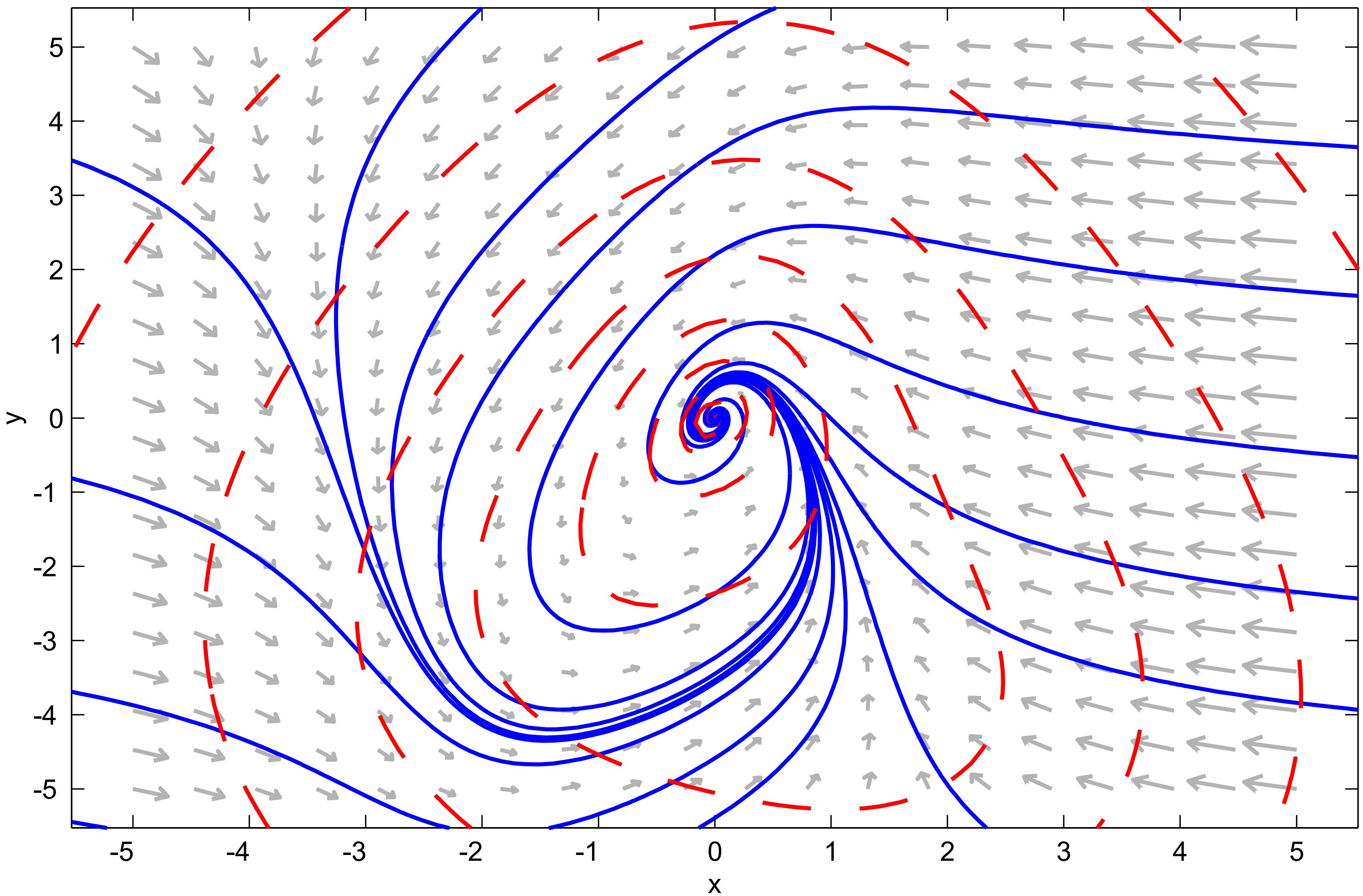
- objective is to minimize the quadratic tracking error

$$\sum_{t=0}^{N-1} \big(x(t) - r(t)\big)^T Q \big(x(t) - r(t)\big)$$

- using binary inputs

$$u_i(t) \in \{-1, 1\} \qquad \text{for all } i = 1, \ldots, m, \text{ and } t = 0, \ldots, N-1$$

Nonlinear Problems: Lyapunov Stability

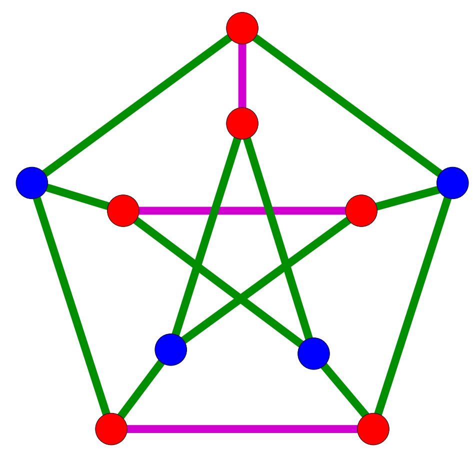


Graph problems

Graph problems appear in many areas: MAX-CUT, independent set, cliques, etc.

MAX CUT partitioning

- Partition the nodes of a graph in two disjoint sets, maximizing the number of edges between sets.
- Practical applications (circuit layout, etc.)
- NP-complete.



How to compute bounds, or exact solutions, for this kind of problems?

polynomial programming

A familiar problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \dots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \dots, p \end{array}$$

objective, inequality and equality constraint functions are all *polynomials*

polynomial nonnegativity

first, consider the case of one inequality; given $f \in \mathbb{R}[x_1, \ldots, x_n]$

$$\boxed{\text{does there exist } x \in \mathbb{R}^n \text{ such that } f(x) < 0}$$

- if not, then f is globally non-negative

$$\boxed{f(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n}$$

  and f is called *positive semidefinite* or *PSD*

- the problem is *NP-hard*, but decidable

- many applications

certificates

the problem

does there exist $x \in \mathbb{R}^n$ such that $f(x) < 0$

- answer yes is easy to verify; exhibit x such that $f(x) < 0$
- answer no is hard; we need a *certificate* or a *witness*
  i.e, a proof that there is no feasible point

Sum of Squares Decomposition

if there are polynomials $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that

$$f(x) = \sum_{i=1}^{s} g_i^2(x)$$

then f is nonnegative

an easily checkable certificate, called a *sum-of-squares (SOS)* decomposition

- how do we find the g_i
- when does such a certificate exist?

example

we can write any polynomial as a *quadratic function of monomials*

$$
\begin{aligned}
f &= 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 \\
&= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7+2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
&= z^T Q(\lambda) z
\end{aligned}
$$

which holds for all $\lambda \in \mathbb{R}$

if for some λ we have $Q(\lambda) \succeq 0$, then we can factorize $Q(\lambda)$

example, continued

e.g., with $\lambda = 6$, we have

$$Q(\lambda) = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

so

$$\begin{aligned} f &= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\ &= \left\| \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy - 3y^2 \end{bmatrix} \right\|^2 \\ &= \left(2xy + y^2\right)^2 + \left(2x^2 + xy - 3y^2\right)^2 \end{aligned}$$

which is an SOS decomposition

sum of squares and semidefinite programming

suppose $f \in \mathbb{R}[x_1, \ldots, x_n]$, of degree $2d$

let z be a vector of all monomials of degree less than or equal to d

f is SOS if and only if there exists Q such that

$$\boxed{\begin{aligned} Q &\succeq 0 \\ f &= z^T Q z \end{aligned}}$$

- this is an SDP in standard primal form
- the number of components of z is $\binom{n+d}{d}$
- comparing terms gives affine constraints on the elements of Q

sum of squares and semidefinite programming

if Q is a feasible point of the SDP, then to construct the SOS representation

factorize $Q = VV^T$, and write $V = \begin{bmatrix} v_1 & \dots v_r \end{bmatrix}$, so that

$$\begin{aligned} f &= z^T V V^T z \\ &= \|V^T z\|^2 \\ &= \sum_{i=1}^{r} (v_i^T z)^2 \end{aligned}$$

- one can factorize using e.g., Cholesky or eigenvalue decomposition
- the number of squares r equals the rank of Q

example

$$
\begin{aligned}
f &= 2x^4 + 2x^3y - x^2y^2 + 5y^4 \\
&= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
&= q_{11}x^4 + 2q_{12}x^3y + (q_{22} + 2q_{13})x^2y^2 + 2q_{23}xy^3 + q_{33}y^4
\end{aligned}
$$

so f is SOS if and only if there exists Q satisfying the SDP

$$
\begin{aligned}
Q \succeq 0 \qquad\qquad q_{11} &= 2 & 2q_{12} &= 2 \\
2q_{12} + q_{22} &= -1 & 2q_{23} &= 0 \\
q_{33} &= 5
\end{aligned}
$$

convexity

the sets of PSD and SOS polynomials are a *convex cones*; i.e.,

$$f, g \text{ PSD} \qquad \Longrightarrow \qquad \lambda f + \mu g \text{ is PSD for all } \lambda, \mu \geq 0$$

let $P_{n,d}$ be the set of PSD polynomials of degree $\leq d$
let $\Sigma_{n,d}$ be the set of SOS polynomials of degree $\leq d$

- both $P_{n,d}$ and $\Sigma_{n,d}$ are *convex cones* in $\mathbb{R}^N$ where $N = \binom{n+d}{d}$
- we know $\Sigma_{n,d} \subset P_{n,d}$, and testing if $f \in P_{n,d}$ is NP-hard
- but testing if $f \in \Sigma_{n,d}$ is an SDP (but a large one)

polynomials in one variable

if $f \in \mathbb{R}[x]$, then f is SOS if and only if f is PSD

example

all real roots must have even multiplicity, and highest coeff. is positive

$$\begin{aligned} f &= x^6 - 10x^5 + 51x^4 - 166x^3 + 342x^2 - 400x + 200 \\ &= (x-2)^2\big(x-(2+i)\big)\big(x-(2-i)\big)\big(x-(1+3i)\big)\big(x-(1-3i)\big) \end{aligned}$$

now reorder complex conjugate roots

$$\begin{aligned} &= (x-2)^2\big(x-(2+i)\big)\big(x-(1+3i)\big)\big(x-(2-i)\big)\big(x-(1-3i)\big) \\ &= (x-2)^2\big((x^2-3x-1)-i(4x-7)\big)\big((x^2-3x-1)+i(4x-7)\big) \\ &= (x-2)^2\big((x^2-3x-1)^2+(4x-7)^2\big) \end{aligned}$$

so every PSD scalar polynomial is the sum of *one or two* squares

quadratic polynomials

a quadratic polynomial in n variables is PSD if and only if it is SOS

because it is PSD if and only if

$$f = x^T Q x$$

where $Q \geq 0$

and it is SOS if and only if

$$\begin{aligned} f &= \sum_i (v_i^T x)^2 \\ &= x^T \Big(\sum_i v_i v_i^T \Big) x \end{aligned}$$

some background

In 1888, Hilbert showed that PSD=SOS if and only if

- $d = 2$, i.e., quadratic polynomials
- $n = 1$, i.e., univariate polynomials
- $d = 4, n = 2$, i.e., quartic polynomials in two variables

$n \backslash d$	2	4	6	8
1	yes	yes	yes	yes
2	yes	*yes*	no	no
3	yes	no	no	no
4	yes	no	no	no

- in general f is PSD does not imply f is SOS

some background

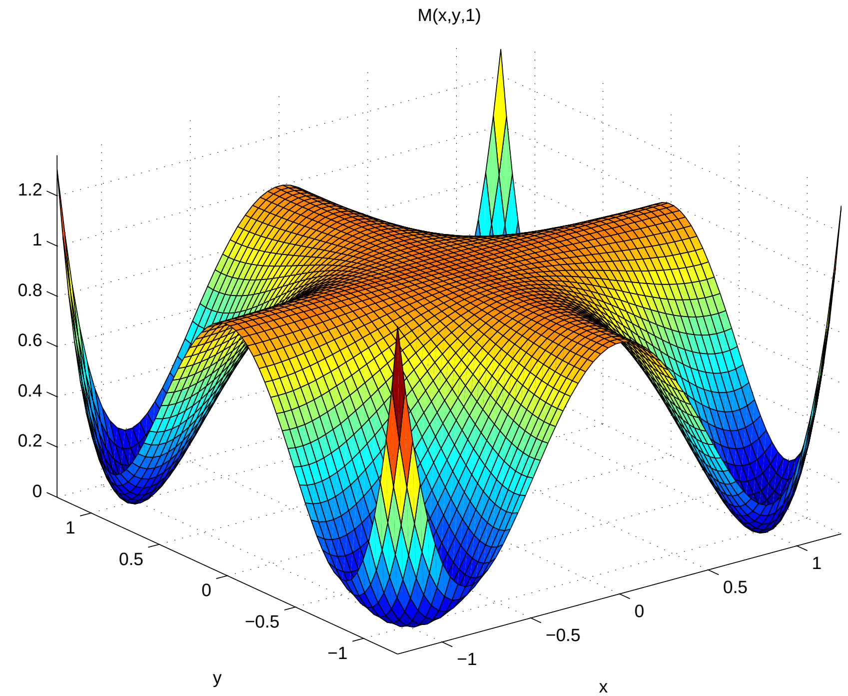
- Connections with Hilbert's 17th problem, solved by Artin: every PSD polynomial is a SOS of *rational functions*.
- If f is not SOS, then can try with gf, for some g.
  - For fixed f, can optimize over g too
  - Otherwise, can use a "universal" construction of Pólya-Reznick.

More about this later.

The Motzkin Polynomial

A positive semidefinite polynomial, that is *not* a sum of squares.

$$M(x, y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$



- Nonnegativity follows from the arithmetic-geometric inequality applied to $(x^2y^4, x^4y^2, 1)$
- Introduce a nonnegative factor $x^2 + y^2 + 1$
- Solving the SDPs we obtain the decomposition:

$$(x^2 + y^2 + 1)\, M(x, y) = (x^2y - y)^2 + (xy^2 - x)^2 + (x^2y^2 - 1)^2 + \\ + \frac{1}{4}(xy^3 - x^3y)^2 + \frac{3}{4}(xy^3 + x^3y - 2xy)^2$$

The Univariate Case:

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2d} x^{2d} \\
&= \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix}^T \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0d} \\ q_{01} & q_{11} & \cdots & q_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ q_{0d} & q_{1d} & \cdots & q_{dd} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix} \\
&= \sum_{i=0}^{d} \Bigg(\sum_{j+k=i} q_{jk} \Bigg) x^i
\end{aligned}
$$

- In the univariate case, the SOS condition is exactly equivalent to non-negativity.
- The matrices A_i in the SDP have a Hankel structure. This can be exploited for efficient computation.

About SOS/SDP

- The resulting SDP problem is polynomially sized (in n, for fixed d).
- By properly choosing the monomials, we can exploit structure (sparsity, symmetries, ideal structure).
- An important feature: the problem is still a SDP *if the coefficients of F are variable*, and the dependence is affine.
- Can optimize over SOS polynomials in affinely described families.

  For instance, if we have $p(x) = p_0(x) + \alpha p_1(x) + \beta p_2(x)$, we can "easily" find values of α, β for which $p(x)$ is SOS.

This fact will be *crucial* in everything that follows. . .

Global Optimization

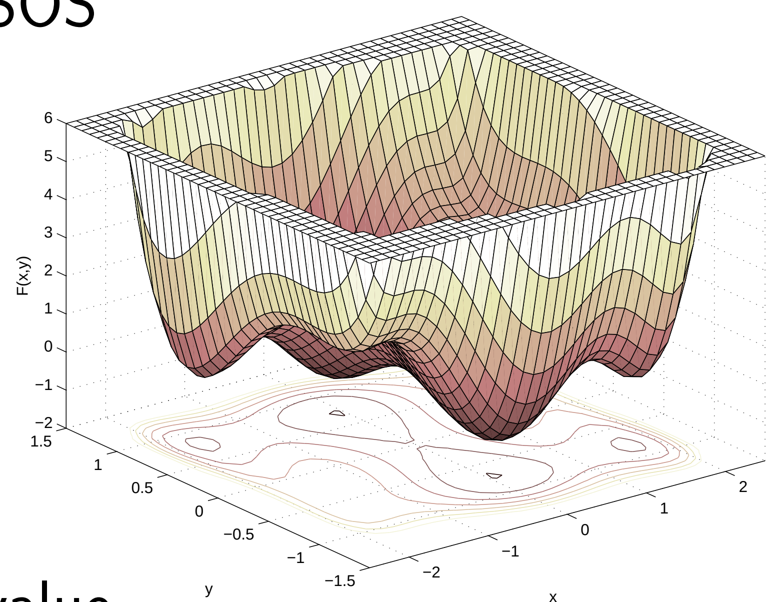
Consider the problem

$$\min_{x,y} f(x,y)$$

with

$$f(x,y) := 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4$$

- Not convex. Many local minima. NP-hard.
- Find the largest γ s.t. $f(x,y) - \gamma$ is SOS
- Essentially due to Shor (1987).
- A semidefinite program (convex!).
- If exact, can recover optimal solution.
- *Surprisingly* effective.



Solving, the maximum γ is -1.0316. Exact value.

Coefficient Space

Let $f_{\alpha\beta}(x) = x^4 + (\alpha + 3\beta)x^3 + 2\beta x^2 - \alpha x + 1$.

What is the set of values of $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha\beta}$ is PSD? SOS?

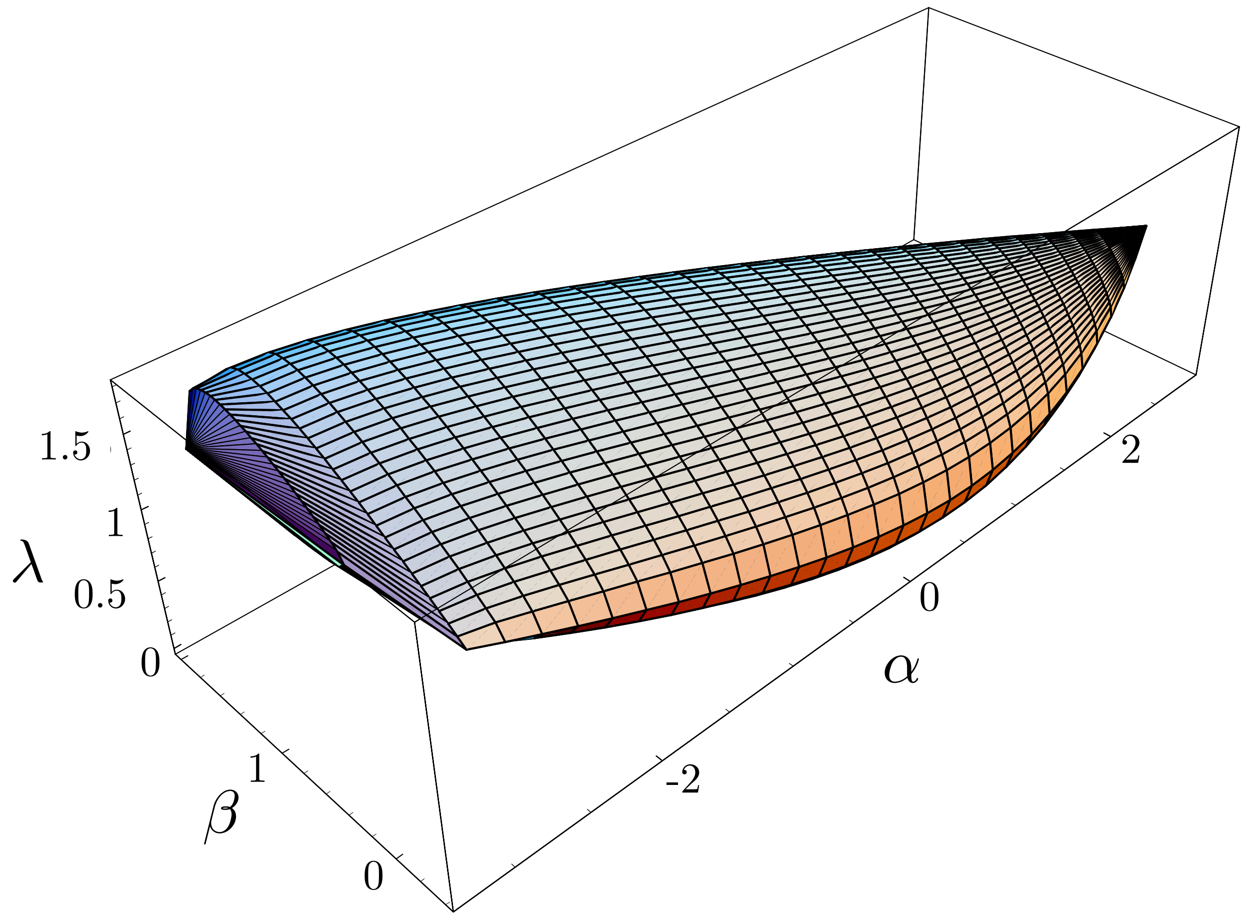
To find a SOS decomposition:

$$\begin{aligned} f_{\alpha,\beta}(x) &= 1 - \alpha x + 2\beta x^2 + (\alpha + 3\beta)x^3 + x^4 \\ &= \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\ &= q_{11} + 2q_{12}x + (q_{22} + 2q_{13})x^2 + 2q_{23}x^3 + q_{33}x^4 \end{aligned}$$

The matrix Q should be PSD and satisfy the affine constraints.

The feasible set is given by:

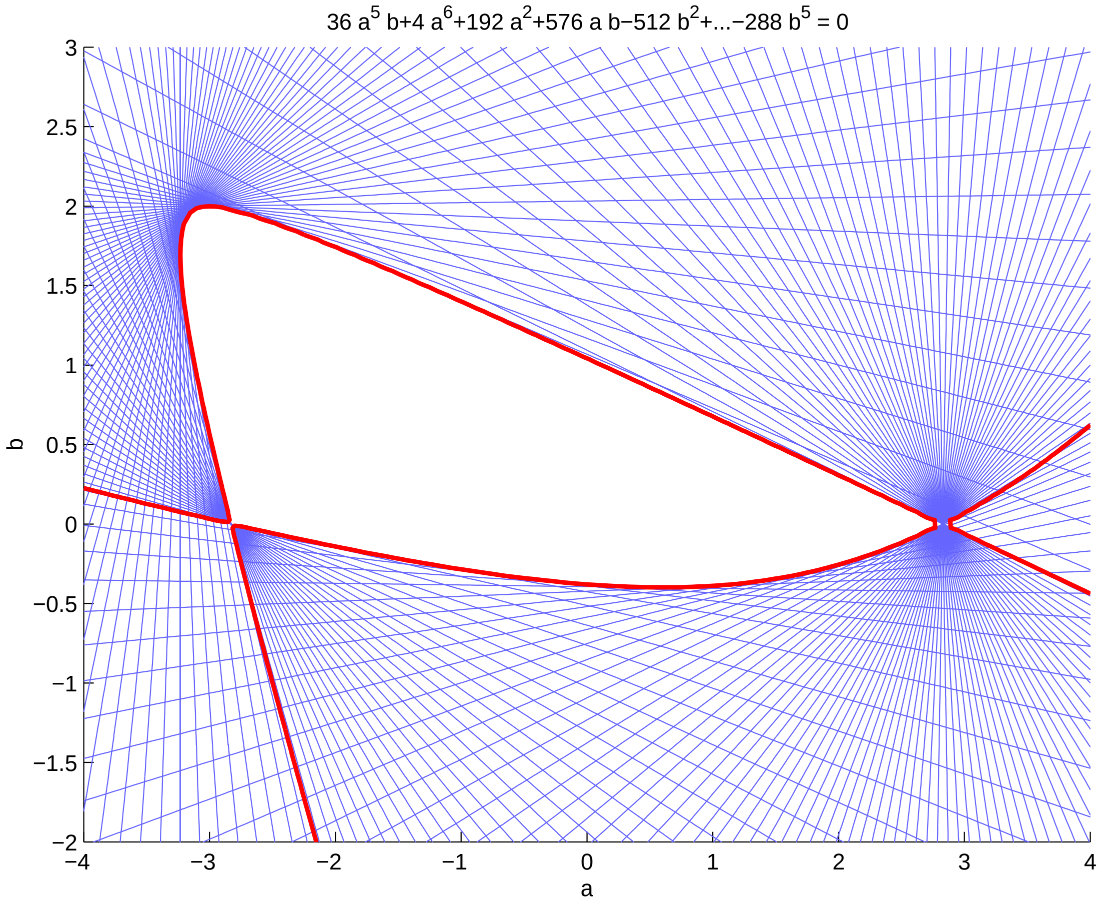
$$\left\{ (\alpha, \beta) \quad \middle| \quad \exists \lambda \text{ s.t. } \begin{bmatrix} 1 & -\frac{1}{2}\alpha & \beta - \lambda \\ -\frac{1}{2}\alpha & 2\lambda & \frac{1}{2}(\alpha + 3\beta) \\ \beta - \lambda & \frac{1}{2}(\alpha + 3\beta) & 1 \end{bmatrix} \succeq 0 \right\}$$



What is the set of values of $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha\beta}$ PSD? SOS?

Recall: in the univariate case PSD=SOS, so here the sets are the same.

- Convex and semialgebraic.
- It is the projection of a spectrahedron in $\mathbb{R}^3$.
- We can easily test membership, or even optimize over it!

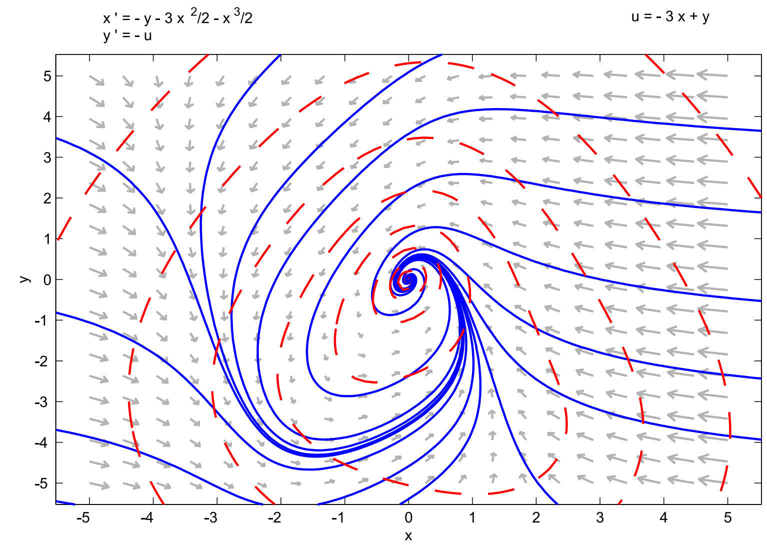


Defined by the curve: $288\beta^5 - 36\alpha^2\beta^4 + 1164\alpha\beta^4 + 1931\beta^4 - 132\alpha^3\beta^3 + 1036\alpha^2\beta^3 + 1956\alpha\beta^3 - 2592\beta^3 - 112\alpha^4\beta^2 + 432\alpha^3\beta^2 + 1192\alpha^2\beta^2 - 1728\alpha\beta^2 + 512\beta^2 - 36\alpha^5\beta + 72\alpha^4\beta + 360\alpha^3\beta - 576\alpha^2\beta - 576\alpha\beta - 4\alpha^6 + 60\alpha^4 - 192\alpha^2 - 256 = 0$

Lyapunov Stability Analysis

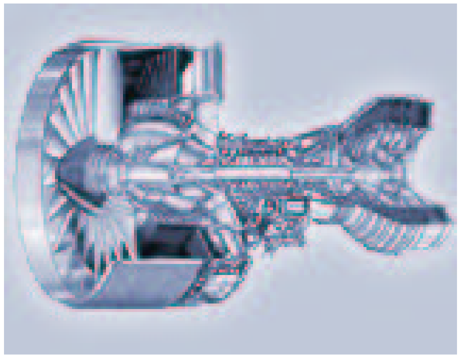
To prove asymptotic stability of $\dot{x} = f(x)$,

$$V(x) > 0 \quad x \neq 0$$
$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)^T f(x) < 0, \quad x \neq 0$$



- For linear systems $\dot{x} = Ax$, quadratic Lyapunov functions $V(x) = x^T P x$
$$P > 0, \qquad A^T P + PA < 0.$$
- With an affine family of candidate polynomial V, $\dot{V}$ is also affine.
- Instead of *checking nonnegativity*, use a *SOS condition*.
- Therefore, for polynomial vector fields and Lyapunov functions, we can check the conditions using the theory described before.

Lyapunov Example



A jet engine model (derived from Moore-Greitzer), with controller:

$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$

Try a generic 4th order polynomial Lyapunov function.

$$V(x,y) = \sum_{0 \le j+k \le 4} c_{jk} x^j y^k$$

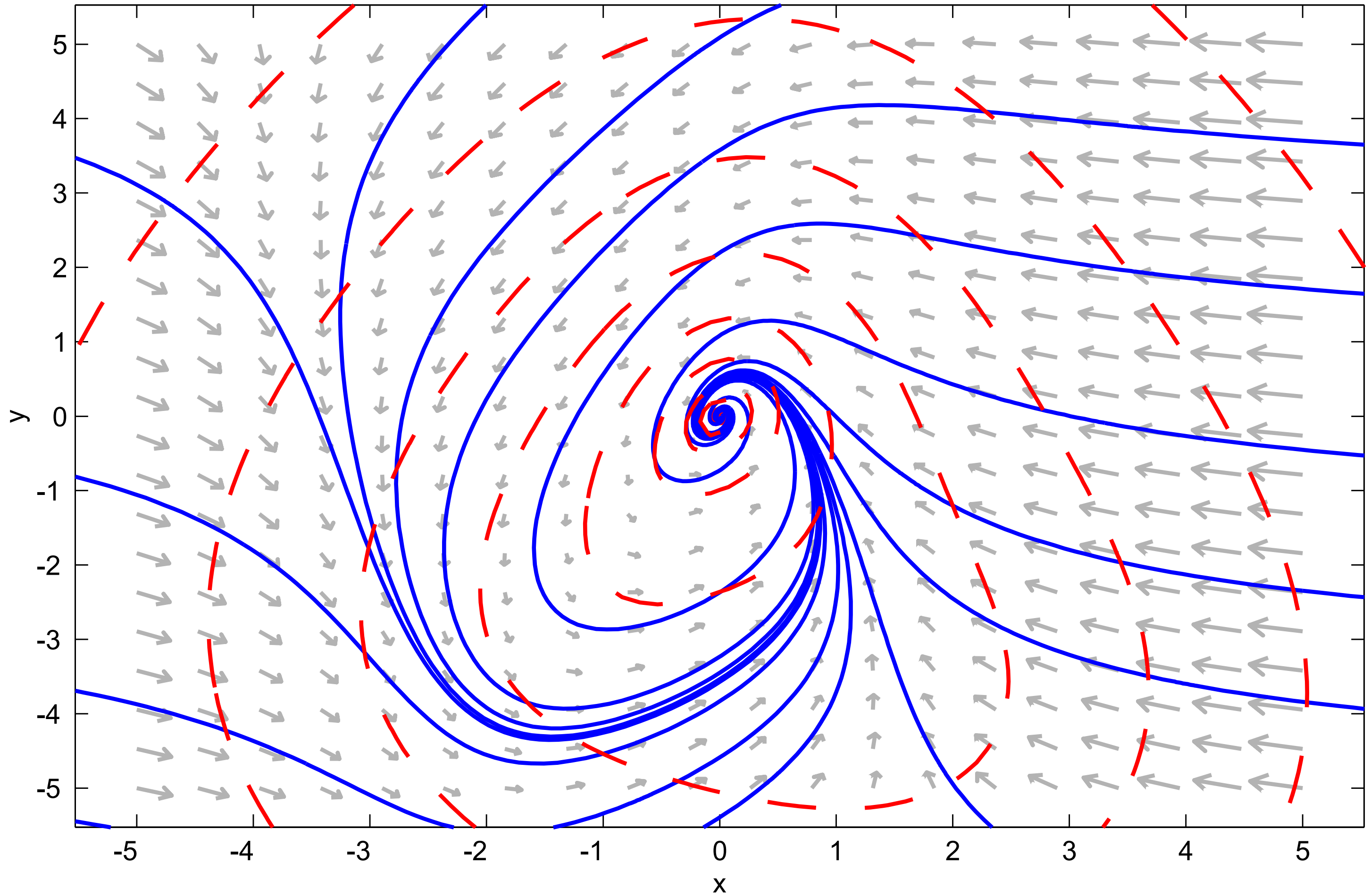
Find a $V(x,y)$ that satisfies the conditions:

- $V(x,y)$ is SOS.
- $-\dot{V}(x,y)$ is SOS.

Both conditions are affine in the c_{jk}. Can do this directly using SOS/SDP!

Lyapunov Example

After solving the SDPs, we obtain a Lyapunov function.



$$V = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2 + 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.0000018868xy^3 + 0.090723y^4$$

Lyapunov Example

Find a Lyapunov function for

$$\begin{aligned}\dot{x} &= -x + (1+x)\,y \\ \dot{y} &= -(1+x)\,x.\end{aligned}$$

we easily find a quartic polynomial

$$V(x,y) = 6x^2 - 2xy + 8y^2 - 2y^3 + 3x^4 + 6x^2y^2 + 3y^4.$$

Both $V(x,y)$ and $(-\dot{V}(x,y))$ are SOS:

$$V(x,y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}, \quad -\dot{V}(x,y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}^T \begin{bmatrix} 10 & 1 & -1 & 1 \\ 1 & 2 & 1 & -2 \\ -1 & 1 & 12 & 0 \\ 1 & -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}$$

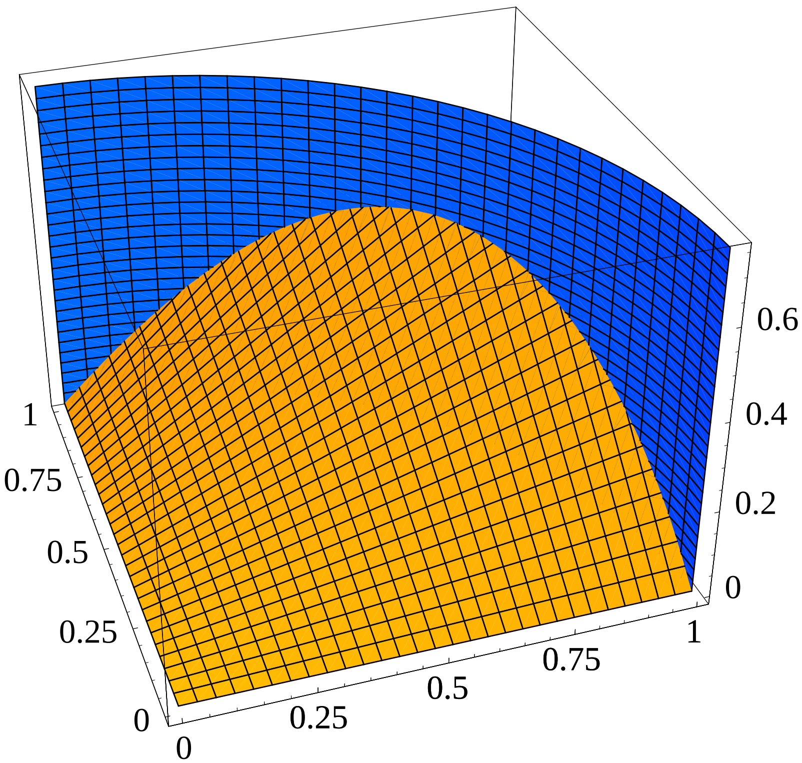
The matrices are positive definite, so this proves asymptotic stability.

Extensions

- Other linear differential inequalities (e.g. Hamilton-Jacobi).
- Many possible variations: nonlinear $\mathcal{H}_\infty$ analysis, parameter dependent Lyapunov functions, etc.
- Can also do local results (for instance, on compact domains).
- Polynomial and rational vector fields, or functions with an underlying algebraic structure.
- Natural extension of the LMIs for the linear case.

Example

$$\begin{array}{ll} \text{minimize} & x_1 x_2 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1^2 + x_2^2 \leq 1 \end{array}$$



- The objective is not convex.
- The Lagrange dual function is

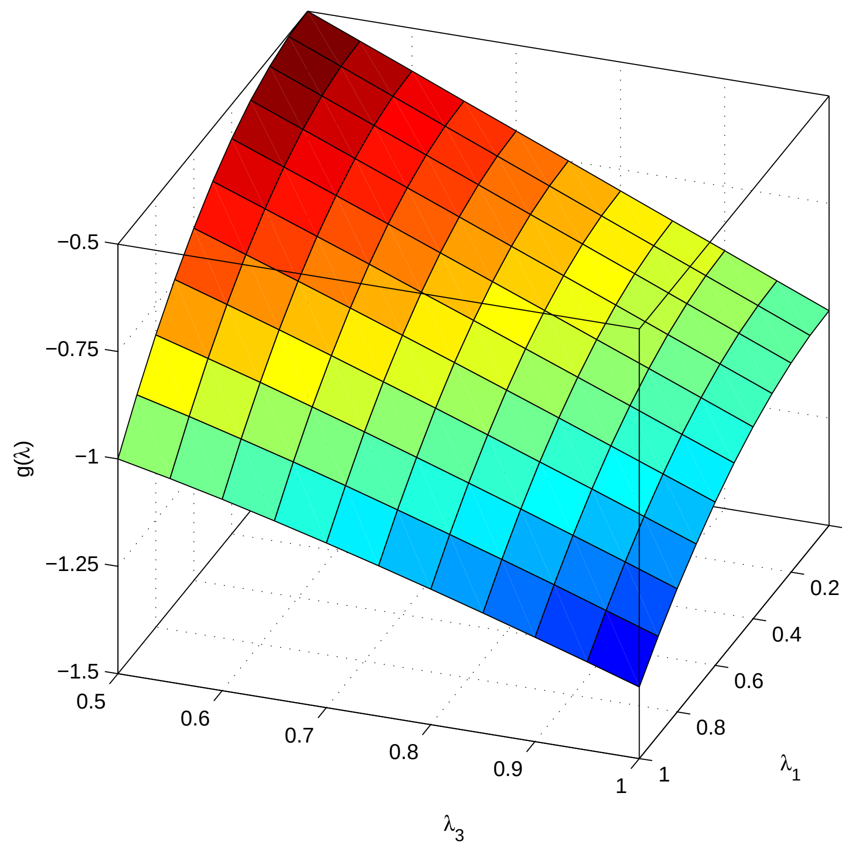
$$g(\lambda) = \inf_x \Big(x_1 x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3 (x_1^2 + x_2^2 - 1) \Big)$$

$$= \begin{cases} -\lambda_3 - \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda_3 & 1 \\ 1 & 2\lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } \lambda_3 > \frac{1}{2} \\ -\infty & \text{otherwise, except bdry} \end{cases}$$

Example, continued

The dual problem is

$$\begin{aligned} \text{maximize} \quad & g(\lambda) \\ \text{subject to} \quad & \lambda_1 \geq 0 \\ & \lambda_2 \geq 0 \\ & \lambda_3 \geq \frac{1}{2} \end{aligned}$$



- By symmetry, if the maximum $g(\lambda)$ is attained, then $\lambda_1 = \lambda_2$ at optimality
- The optimal $g(\lambda^\star) = -\frac{1}{2}$ at $\lambda^\star = (0, 0, \frac{1}{2})$
- Here we see an example of a *duality gap*; the primal optimal is strictly greater than the dual optimal

Example, continued

It turns out that, using the Schur complement, the dual problem may be written as

$$
\begin{aligned}
\text{maximize} \quad & \gamma \\
\text{subject to} \quad & \begin{bmatrix} -2\gamma - 2\lambda_3 & \lambda_1 & \lambda_2 \\ \lambda_1 & 2\lambda_3 & 1 \\ \lambda_2 & 1 & 2\lambda_3 \end{bmatrix} > 0 \\
& \lambda_1 > 0 \\
& \lambda_2 > 0
\end{aligned}
$$

We'll see a systematic way to convert a dual problem to an SDP, whenever the objective and constraint functions are polynomials.

The Dual is Not Intrinsic

- The dual problem, and its corresponding optimal value, are *not properties of the primal feasible set and objective function alone*.
- Instead, they depend on the particular equations and inequalities used

To construct equivalent primal optimization problems with different duals:

- replace the objective $f_0(x)$ by $h(f_0(x))$ where h is increasing
- introduce new variables and associated constraints, e.g.

$$\text{minimize} \quad (x_1 - x_2)^2 + (x_2 - x_3)^2$$

  is replaced by

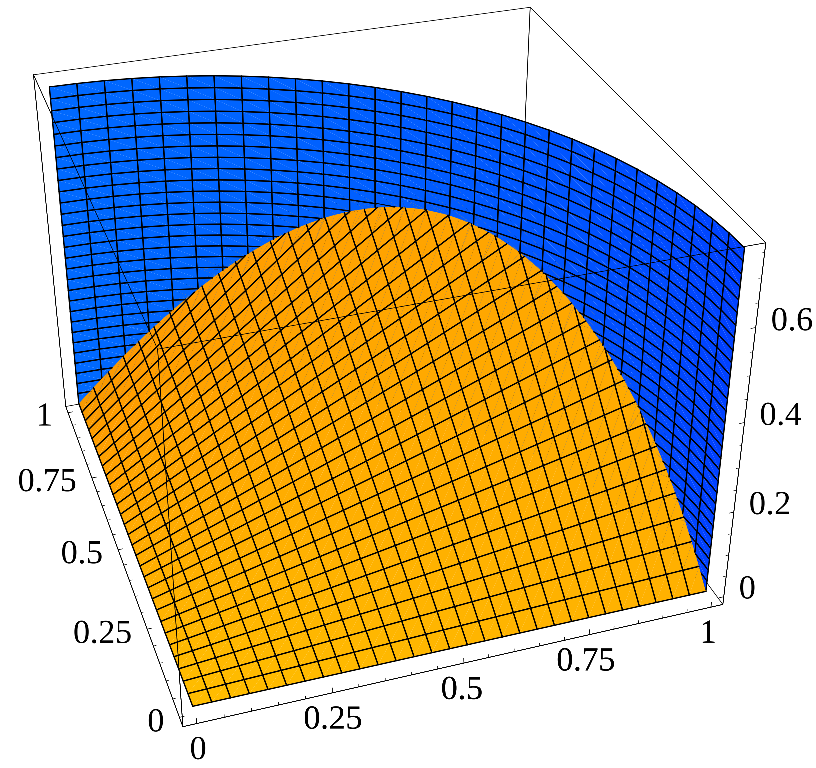
$$\begin{array}{ll} \text{minimize} & (x_1 - x_2)^2 + (x_4 - x_3)^2 \\ \text{subject to} & x_2 = x_4 \end{array}$$

- add redundant constraints

Example

Adding the redundant constraint $x_1 x_2 \geq 0$ to the previous example gives

$$\begin{aligned} \text{minimize} \quad & x_1 x_2 \\ \text{subject to} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1^2 + x_2^2 \leq 1 \\ & x_1 x_2 \geq 0 \end{aligned}$$



Clearly, this has the same primal feasible set and same optimal value as before

Example Continued

The Lagrange dual function is

$$g(\lambda) = \inf_x \Big(x_1 x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3 (x_1^2 + x_2^2 - 1) - \lambda_4 x_1 x_2 \Big)$$

$$= \begin{cases} -\lambda_3 - \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda_3 & 1-\lambda_4 \\ 1-\lambda_4 & 2\lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & \text{if } 2\lambda_3 > 1 - \lambda_4 \\ -\infty & \text{otherwise, except bdry} \end{cases}$$

- Again, this problem may also be written as an SDP. The optimal value is $g(\lambda^\star) = 0$ at $\lambda^\star = (0, 0, 0, 1)$
- Adding redundant constraints makes the dual bound *tighter*
- This always happens! Such redundant constraints are called *valid inequalities*.

Constructing Valid Inequalities

The function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *valid inequality* if

$$f(x) \geq 0 \qquad \text{for all feasible } x$$

Given a set of inequality constraints, we can generate others as follows.

(i) If f_1 and f_2 define valid inequalities, then so does $h(x) = f_1(x) + f_2(x)$

(ii) If f_1 and f_2 define valid inequalities, then so does $h(x) = f_1(x) f_2(x)$

(iii) For any f, the function $h(x) = f(x)^2$ defines a valid inequality

Now we can use *algebra* to generate valid inequalities.

Valid Inequalities and Cones

- The set of *polynomial* functions on $\mathbb{R}^n$ with real coefficients is denoted $\mathbb{R}[x_1, \ldots, x_n]$
- Computationally, they are easy to *parametrize*. We will consider polynomial constraint functions.

A set of polynomials $P \subset \mathbb{R}[x_1, \ldots, x_n]$ is called a *cone* if

(i) $f_1 \in P$ and $f_2 \in P$ implies $f_1 f_2 \in P$

(ii) $f_1 \in P$ and $f_2 \in P$ implies $f_1 + f_2 \in P$

(iii) $f \in \mathbb{R}[x_1, \ldots, x_n]$ implies $f^2 \in P$

It is called a *proper cone* if $-1 \notin P$

By applying the above rules to the inequality constraint functions, we can generate a *cone of valid inequalities*

Algebraic Geometry

- There is a correspondence between the *geometric object* (the feasible subset of $\mathbb{R}^n$) and the *algebraic object* (the cone of valid inequalities)
- This is a *dual* relationship; we'll see more of this later.
- The dual problem is constructed from the *cone*.
- For equality constraints, there is another algebraic object; the *ideal* generated by the equality constraints.
- For optimization, we need to look both at the geometric objects (for the primal) and the algebraic objects (for the dual problem)

Cones

- For $S \subset \mathbb{R}^n$, the cone defined by S is

$$\mathcal{C}(S) = \left\{ f \in \mathbb{R}[x_1, \dots, x_n] \;\middle|\; f(x) \geq 0 \text{ for all } x \in S \right\}$$

- If P_1 and P_2 are cones, then so is $P_1 \cap P_2$

- A polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is called a *sum-of-squares* (SOS) if

$$f(x) = \sum_{i=1}^{r} s_i(x)^2$$

for some polynomials $s_1, \dots, s_r$ and some $r \geq 0$. The set of SOS polynomials Σ is a cone.

- Every cone contains Σ.

Cones

The set $\mathbf{monoid}\{f_1, \dots, f_m\} \subset \mathbb{R}[x_1, \dots, x_n]$ is the set of all finite products of polynomials f_i, together with 1.

The smallest cone containing the polynomials $f_1, \dots, f_m$ is

$$\mathbf{cone}\{f_1, \dots, f_m\} = \left\{ \sum_{i=1}^{r} s_i g_i \mid s_0, \dots, s_r \in \Sigma, \right.$$
$$\left. g_i \in \mathbf{monoid}\{f_1, \dots, f_m\} \right\}$$

$\mathbf{cone}\{f_1, \dots, f_m\}$ is called the *cone generated by* $f_1, \dots, f_m$

Explicit Parametrization of the Cone

- If $f_1, \ldots, f_m$ are valid inequalities, then so is every polynomial in $\mathbf{cone}\{f_1, \ldots, f_m\}$

- The polynomial h is an element of $\mathbf{cone}\{f_1, \ldots, f_m\}$ if and only if

$$h(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) f_i(x) + \sum_{i \neq j} r_{ij}(x) f_i(x) f_j(x) + \ldots$$

where the s_i and r_{ij} are *sums-of-squares*.

An Algebraic Approach to Duality

Suppose $f_1, \ldots, f_m$ are polynomials, and consider the feasibility problem

does there exist $x \in \mathbb{R}^n$ such that
$f_i(x) \geq 0 \qquad \text{for all } i = 1, \ldots, m$

Every polynomial in $\mathbf{cone}\{f_1, \ldots, f_m\}$ is non-negative on the feasible set.

So if there is a polynomial $q \in \mathbf{cone}\{f_1, \ldots, f_m\}$ which satisfies

$$q(x) \leq -\varepsilon < 0 \qquad \text{for all } x \in \mathbb{R}^n$$

then the primal problem is infeasible.

Example

Let's look at the feasibility version of the previous problem. Given $t \in \mathbb{R}$, does there exist $x \in \mathbb{R}^2$ such that

$$\begin{aligned} x_1 x_2 &\le t \\ x_1^2 + x_2^2 &\le 1 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Equivalently, is the set S nonempty, where

$$S = \left\{ x \in \mathbb{R}^n \mid f_i(x) \ge 0 \text{ for all } i = 1, \dots, m \right\}$$

where

$$f_1(x) = t - x_1 x_2 \qquad f_2(x) = 1 - x_1^2 - x_2^2$$
$$f_3(x) = x_1 \qquad f_4(x) = x_2$$

Example Continued

Let $q(x) = f_1(x) + \frac{1}{2}f_2(x)$. Then clearly $q \in \mathbf{cone}\{f_1, f_2, f_3, f_4\}$ and

$$\begin{aligned} q(x) &= t - x_1x_2 + \tfrac{1}{2}(1 - x_1^2 - x_2^2) \\ &= t + \tfrac{1}{2} - \tfrac{1}{2}(x_1 + x_2)^2 \\ &\leq t + \tfrac{1}{2} \end{aligned}$$

So for any $t < -\frac{1}{2}$, the primal problem is infeasible.

This corresponds to Lagrange multipliers $(1, \frac{1}{2})$ for the thm. of alternatives.

Alternatively, this is a proof by contradiction.

- If there exists x such that $f_i(x) \geq 0$ for $i = 1, \ldots, 4$ then we must also have $q(x) \geq 0$, since $q \in \mathbf{cone}\{f_1, \ldots, f_4\}$
- But we proved that q is negative if $t < -\frac{1}{2}$

Example Continued

We can also do better by using other functions in the cone. Try

$$\begin{aligned} q(x) &= f_1(x) + f_3(x)f_4(x) \\ &= t \end{aligned}$$

giving the stronger result that for any $t < 0$ the inequalities are infeasible. Again, this corresponds to Lagrange multipliers $(1, 1)$

- In both of these examples, we found q in the cone which was globally negative. We can view q as the Lagrangian function evaluated at a particular value of λ
- The Lagrange multiplier procedure is *searching* over a *particular subset* of functions in the cone; those which are generated by *linear combinations* of the original constraints.
- By searching over more functions in the cone we can do better

Normalization

In the above example, we have

$$q(x) = t + \tfrac{1}{2} - \tfrac{1}{2}(x_1 + x_2)^2$$

We can also show that $-1 \in \mathbf{cone}\{f_1, \ldots, f_4\}$, which gives a very simple proof of primal infeasibility.

Because, for $t < -\frac{1}{2}$, we have

$$-1 = a_0 q(x) + a_1 (x_1 + x_2)^2$$

and by construction q is in the cone, and $(x_1 + x_2)^2$ is a sum of squares.

Here a_0 and a_1 are positive constants

$$a_0 = \frac{-2}{2t+1} \qquad a_1 = \frac{-1}{2t+1}$$

An Algebraic Dual Problem

Suppose $f_1, \ldots, f_m$ are polynomials. The primal feasibility problem is

does there exist $x \in \mathbb{R}^n$ such that
$$f_i(x) \geq 0 \qquad \text{for all } i = 1, \ldots, m$$

The *dual feasibility problem* is

Is it true that $-1 \in \mathbf{cone}\{f_1, \ldots, f_m\}$

If the dual problem is feasible, then the primal problem is infeasible.

In fact, a result called the *Positivstellensatz* implies that *strong duality* holds here.

Interpretation: Searching the Cone

- Lagrange duality is searching over *linear combinations* with nonnegative coefficients

$$\lambda_1 f_1 + \cdots + \lambda_m f_m$$

to find a globally negative function as a certificate

- The above algebraic procedure is searching over *conic combinations*

$$s_0(x) + \sum_{i=1}^{m} s_i(x) f_i(x) + \sum_{i \neq j} r_{ij}(x) f_i(x) f_j(x) + \dots$$

where the s_i and r_{ij} are *sums-of-squares*

Interpretation: Formal Proof

We can also view this as a type of *formal proof*:

- View $f_1, \ldots, f_m$ are *predicates*, with $f_i(x) \geq 0$ meaning that x satisfies f_i.
- Then $\mathbf{cone}\{f_1, \ldots, f_m\}$ consists of predicates which are *logical consequences* of $f_1, \ldots, f_m$.
- If we find -1 in the cone, then we have a proof by contradiction.

Our objective is to *automatically search* the cone for negative functions; i.e., proofs of infeasibility.

Example: Linear Inequalities

Does there exist $x \in \mathbb{R}^n$ such that

$$Ax \geq 0$$

$$c^T x \leq -1$$

Write A in terms of its rows $A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$,

then we have inequality constraints defined by linear polynomials

$$f_i(x) = a_i^T x \qquad \text{for } i = 1, \dots, m$$

$$f_{m+1}(x) = -1 - c^T x$$

Example: Linear Inequalities

We'll search over functions $q \in \textbf{cone}\{f_1, \ldots, f_{m+1}\}$ of the form

$$q(x) = \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_{m+1}(x)$$

Then the algebraic form of the dual is:

> does there exist $\lambda_i \geq 0$, $\mu \geq 0$ such that
> $$q(x) = -1 \qquad \text{for all } x$$

if the dual is feasible, then the primal problem is infeasible

Example: Linear Inequalities

The above dual condition is

$$\lambda^T A x + \mu(-1 - c^T x) = -1 \qquad \text{for all } x$$

which holds if and only if $A^T \lambda = \mu c$ and $\mu = 1$.

So we can state the duality result as follows.

Farkas Lemma

If there exists $\lambda \in \mathbb{R}^m$ such that

$$A^T \lambda = c \qquad \text{and} \qquad \lambda \geq 0$$

then there does not exist $x \in \mathbb{R}^n$ such that

$$Ax \geq 0 \qquad \text{and} \qquad c^T x \leq -1$$

Farkas Lemma

Farkas Lemma states that the following are strong alternatives

(i) there exists $\lambda \in \mathbb{R}^m$ such that $A^T\lambda = c$ and $\lambda \geq 0$

(ii) there exists $x \in \mathbb{R}^n$ such that $Ax \geq 0$ and $c^T x < 0$

Since this is just Lagrangian duality, there is a geometric interpretation

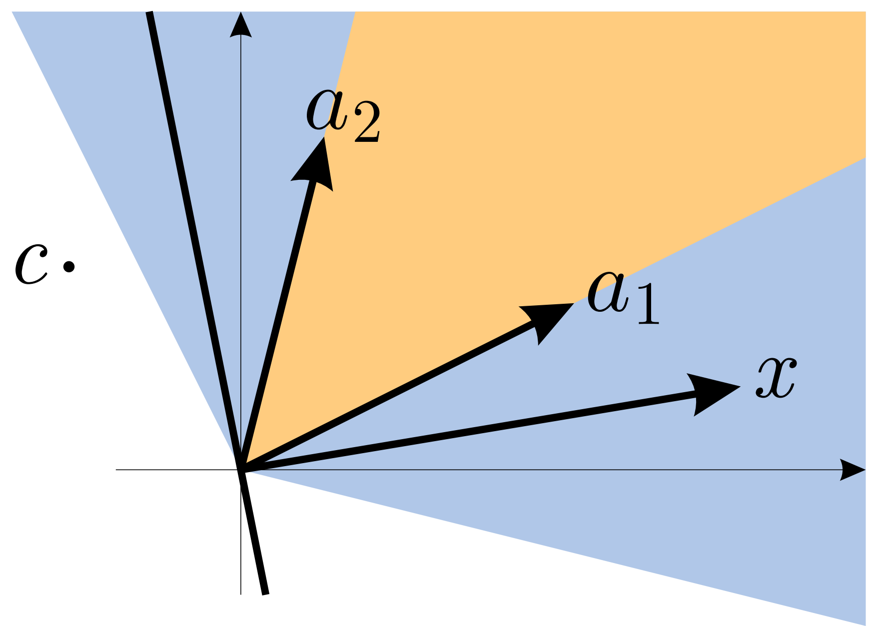
(i) c is in the convex cone

$$\{ A^T\lambda \mid \lambda \geq 0 \}$$

(ii) x defines the hyperplane

$$\{ y \in \mathbb{R}^n \mid y^T x = 0 \}$$

which separates c from the cone



Optimization Problems

Let's return to optimization problems instead of feasibility problems.

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \geq 0 \qquad \text{for all } i = 1, \dots, m \end{aligned}$$

The corresponding feasibility problem is

$$\begin{aligned} t - f_0(x) &\geq 0 \\ f_i(x) &\geq 0 \qquad \text{for all } i = 1, \dots, m \end{aligned}$$

One simple dual is to find polynomials s_i and r_{ij} such that

$$t - f_0(x) + \sum_{i=1}^{m} s_i(x) f_i(x) + \sum_{i \neq j} r_{ij}(x) f_i(x) f_j(x) + \dots$$

is globally negative, where the s_i and r_{ij} are *sums-of-squares*

Optimization Problems

We can combine this with a maximization over t

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & t - f_0(x) + \sum_{i=1}^{m} s_i(x) f_i(x) + \\ & \qquad \sum_{i=1}^{m} \sum_{j=1}^{m} r_{ij}(x) f_i(x) f_j(x) \leq 0 \text{ for all } x \\ & s_i, r_{ij} \text{ are sums-of-squares} \end{array}$$

- The variables here are (coefficients of) the polynomials s_i, r_i
- We will see later how to approach this kind of problem using semidefinite programming

equality constraints

consider the feasibility problem

does there exist $x \in \mathbb{R}^n$ such that
$f_i(x) = 0 \qquad$ for all $i = 1, \dots, m$

the function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *valid equality constraint* if

$$f(x) = 0 \qquad \text{for all feasible } x$$

given a set of equality constraints, we can generate others as follows

(i) if f_1 and f_2 are valid equalities, then so is $f_1 + f_2$

(ii) for any $h \in \mathbb{R}[x_1, \dots, x_n]$, if f is a valid equality, then so is hf

using these will make the dual bound *tighter*

ideals and valid equality constraints

a set of polynomials $I \subset \mathbb{R}[x_1, \ldots, x_n]$ is called an *ideal* if

(i) $f_1 + f_2 \in I$ for all $f_1, f_2 \in I$

(ii) $fh \in I$ for all $f \in I$ and $h \in \mathbb{R}[x_1, \ldots, x_n]$

- given $f_1, \ldots, f_m$, we can generate an *ideal of valid equalities* by repeatedly applying these rules
- this gives the *ideal generated by* $f_1, \ldots, f_m$,

$$\mathbf{ideal}\{f_1, \ldots, f_m\} = \left\{ \sum_{i=1}^{m} h_i f_i \mid h_i \in \mathbb{R}[x_1, \ldots, x_n] \right\}$$

written $\mathbf{ideal}\{f_1, \ldots, f_m\}$, or sometimes $\langle f_1, \ldots, f_m \rangle$.

generators of an ideal

- every polynomial in $\mathbf{ideal}\{f_1, \ldots, f_m\}$ is a valid equality.
- $\mathbf{ideal}\{f_1, \ldots, f_m\}$ is the smallest ideal containing $f_1, \ldots, f_m$.
- the polynomials $f_1, \ldots, f_m$ are called the *generators*, or a *basis*, of the ideal.

properties of ideals

- if I_1 and I_2 are ideals, then so is $I_1 \cap I_2$
- an ideal generated by one polynomial is called a *principal ideal*

Feasibility of Semialgebraic Sets

Suppose S is a *semialgebraic set* represented by polynomial inequalities and equations

$$S = \left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0,\ h_j(x) = 0 \text{ for all } i = 1, \ldots, m,\ j = 1, \ldots, p \right\}$$

we'd like to solve the feasibility problem

Is S non-empty?

- Important, non-trivial result: the feasibility problem is *decidable*.
- But NP-hard (even for a single polynomial, as we have seen)
- We would like to *certify* infeasibility

The Real Nullstellensatz

Recall Σ is the cone of polynomials representable as *sums of squares*.

Suppose $h_1, \dots, h_m \in \mathbb{R}[x_1, \dots, x_n]$.

$$-1 \in \Sigma + \textbf{ideal}\{h_1, \dots, h_m\} \qquad \Longleftrightarrow \qquad \mathcal{V}_{\mathbb{R}}\{h_1, \dots, h_m\} = \emptyset$$

Equivalently, there is no $x \in \mathbb{R}^n$ such that

$$h_i(x) = 0 \qquad \text{for all } i = 1, \dots, m$$

if and only if there exists $t_1, \dots, t_m \in \mathbb{R}[x_1, \dots, x_n]$ and $s \in \Sigma$ such that

$$-1 = s + t_1 h_1 + \cdots + t_m h_m$$

Example

Suppose $h(x) = x^2 + 1$. Then clearly $\mathcal{V}_{\mathbb{R}}\{h\} = \emptyset$

We saw earlier that the complex Nullstellensatz cannot be used to prove emptyness of $\mathcal{V}_{\mathbb{R}}\{h\}$

But we have

$$-1 = s + th$$

with

$$s(x) = x^2 \qquad \text{and} \qquad t(x) = -1$$

and so the real Nullstellensatz implies $\mathcal{V}_{\mathbb{R}}\{h\} = \emptyset$.

The polynomial equation $-1 = s + th$ gives a certificate of infeasibility.

The Positivstellensatz

We now turn to feasibility for *basic semialgebraic sets*, with primal problem

Does there exist $x \in \mathbb{R}^n$ such that

$$\begin{aligned} f_i(x) &\geq 0 \qquad \text{for all } i = 1, \dots, m \\ h_j(x) &= 0 \qquad \text{for all } j = 1, \dots, p \end{aligned}$$

Call the feasible set S; recall

- every polynomial in $\mathbf{cone}\{f_1, \dots, f_m\}$ is nonnegative on S
- every polynomial in $\mathbf{ideal}\{h_1, \dots, h_p\}$ is zero on S

The *Positivstellensatz* (Stengle 1974)

$$S = \emptyset \quad \iff \quad -1 \in \mathbf{cone}\{f_1, \dots, f_m\} + \mathbf{ideal}\{h_1, \dots, h_m\}$$

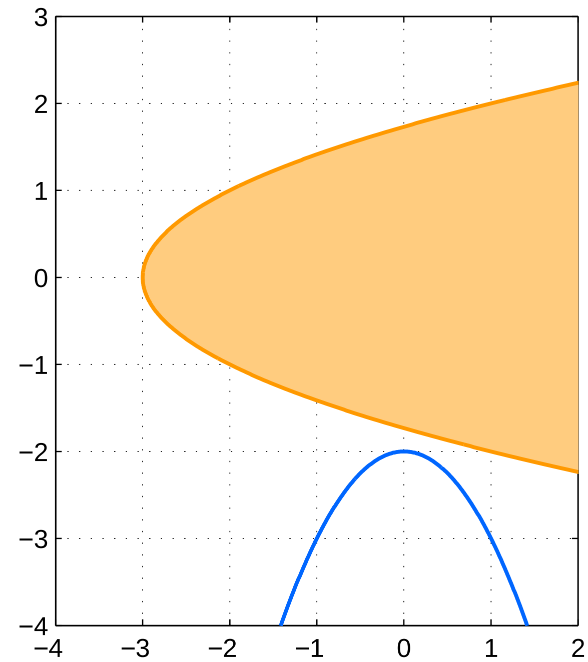
Example

Consider the feasibility problem

$$S = \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) \geq 0, h(x,y) = 0 \}$$

where

$$f(x,y) = x - y^2 + 3$$
$$h(x,y) = y + x^2 + 2$$



By the P-satz, the primal is infeasible if and only if there exist polynomials $s_1, s_2 \in \Sigma$ and $t \in \mathbb{R}[x,y]$ such that

$$-1 = s_1 + s_2 f + th$$

A certificate is given by

$$s_1 = \tfrac{1}{3} + 2\left(y + \tfrac{3}{2}\right)^2 + 6\left(x - \tfrac{1}{6}\right)^2, \quad s_2 = 2, \quad t = -6.$$

Explicit Formulation of the Positivstellensatz

The primal problem is

Does there exist $x \in \mathbb{R}^n$ such that

$$f_i(x) \geq 0 \qquad \text{for all } i = 1, \dots, m$$

$$h_j(x) = 0 \qquad \text{for all } j = 1, \dots, p$$

The dual problem is

Do there exist $t_i \in \mathbb{R}[x_1, \dots, x_n]$ and $s_i, r_{ij}, \dots \in \Sigma$ such that

$$-1 = \sum_i h_i t_i + s_0 + \sum_i s_i f_i + \sum_{i \neq j} r_{ij} f_i f_j + \cdots$$

These are *strong alternatives*

Testing the Positivstellensatz

Do there exist $t_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $s_i, r_{ij}, \ldots \in \Sigma$ such that

$$-1 = \sum_i t_i h_i + s_0 + \sum_i s_i f_i + \sum_{i \neq j} r_{ij} f_i f_j + \cdots$$

- This is a convex feasibility problem in $t_i, s_i, r_{ij}, \ldots$
- To solve it, we need to choose a subset of the cone to search; i.e., the maximum degree of the above polynomial; then the problem is a *semidefinite program*
- This gives a *hierarchy* of syntactically verifiable certificates
- The validity of a certificate may be easily checked; e.g., linear algebra, random sampling
- Unless NP=co-NP, the certificates cannot *always* be polynomially sized.

Example: Farkas Lemma

The primal problem; does there exist $x \in \mathbb{R}^n$ such that

$$\boxed{Ax + b \geq 0 \qquad Cx + d = 0}$$

Let $f_i(x) = a_i^T x + b_i$, $h_i(x) = c_i^T x + d_i$. Then this system is infeasible if and only if

$$-1 \in \mathbf{cone}\{f_1, \dots, f_m\} + \mathbf{ideal}\{h_1, \dots, h_p\}$$

Searching over *linear combinations*, the primal is infeasible if there exist $\lambda \geq 0$ and μ such that

$$\lambda^T(Ax + b) + \mu^T(Cx + d) = -1$$

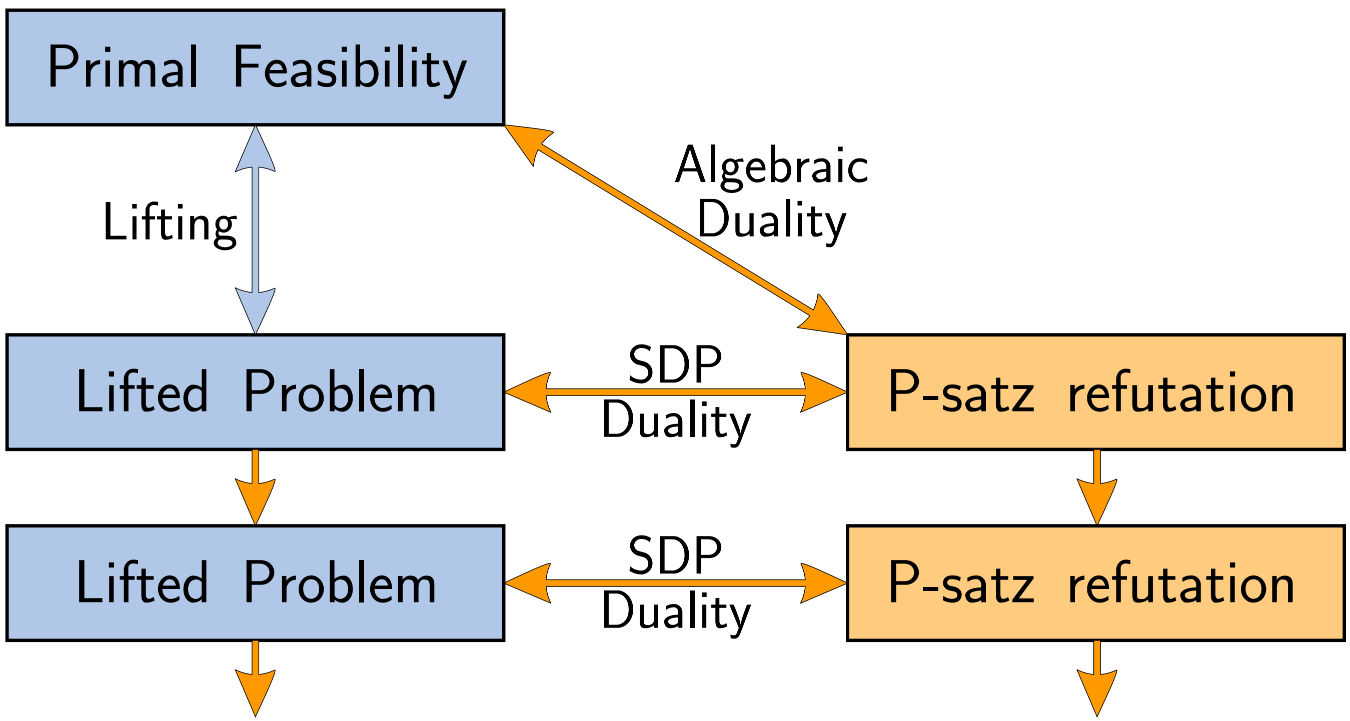
Equating coefficients, this is equivalent to

$$\boxed{\lambda^T A + \mu^T C = 0 \quad \lambda^T b + \mu^T d = -1 \quad \lambda \geq 0}$$

Hierarchy of Certificates

- Interesting connections with logic, proof systems, etc.
- Failure to prove infeasibility (may) provide points in the set.
- Tons of applications:
  optimization, copositivity, dynamical systems, quantum mechanics...

General Scheme



Special Cases

Many known methods can be interpreted as fragments of P-satz refutations.

- LP duality: linear inequalities, constant multipliers.
- S-procedure: quadratic inequalities, constant multipliers
- Standard SDP relaxations for QP.
- The *linear representations* approach for functions f strictly positive on the set defined by $f_i(x) \geq 0$.

$$f(x) = s_0 + s_1 f_1 + \cdots + s_n f_n, \qquad s_i \in \Sigma$$

Converse Results

- *Losslessness:* when can we restrict *a priori* the class of certificates?
- Some cases are known; e.g., additional conditions such as linearity, perfect graphs, compactness, finite dimensionality, etc, can ensure specific *a priori* properties.

Example: Boolean Minimization

$$x^T Q x \leq \gamma$$
$$x_i^2 - 1 = 0$$

A P-satz refutation holds if there is $S \succeq 0$ and $\lambda \in \mathbb{R}^n$, $\varepsilon > 0$ such that

$$-\varepsilon = x^T S x + \gamma - x^T Q x + \sum_{i=1}^{n} \lambda_i (x_i^2 - 1)$$

which holds if and only if there exists a diagonal Λ such that $Q \succeq \Lambda$, $\gamma = \mathbf{trace}\,\Lambda - \varepsilon$.

The corresponding optimization problem is

$$\begin{array}{ll} \text{maximize} & \mathbf{trace}\,\Lambda \\ \text{subject to} & Q \succeq \Lambda \\ & \Lambda \text{ is diagonal} \end{array}$$

Example: S-Procedure

The primal problem; does there exist $x \in \mathbb{R}^n$ such that

$$x^T F_1 x \geq 0$$
$$x^T F_2 x \geq 0$$
$$x^T x = 1$$

We have a P-satz refutation if there exists $\lambda_1, \lambda_2 \geq 0$, $\mu \in \mathbb{R}$ and $S \succeq 0$ such that

$$-1 = x^T S x + \lambda_1 x^T F_1 x + \lambda_2 x^T F_2 x + \mu(1 - x^T x)$$

which holds if and only if there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\lambda_1 F_1 + \lambda_2 F_2 \leq -I$$

Subject to an additional mild constraint qualification, this condition is also *necessary* for infeasibility.

Exploiting Structure

What algebraic properties of the polynomial system yield efficient computation?

- *Sparseness:* few nonzero coefficients.
  - Newton polytopes techniques
  - Complexity does not depend on the degree
- *Symmetries:* invariance under a transformation group
  - Frequent in practice. Enabling factor in applications.
  - Can reflect underlying physical symmetries, or modelling choices.
  - SOS on *invariant rings*
  - Representation theory and invariant-theoretic techniques.
- *Ideal structure:* Equality constraints.
  - SOS on *quotient rings*
  - Compute in the coordinate ring. Quotient bases (Groebner)

Example: Structured Singular Value

- Structured singular value μ and related problems: provides better upper bounds.
- μ is a measure of robustness: how big can a structured perturbation be, without losing stability.
- A standard semidefinite relaxation: the μ upper bound.
  - Morton and Doyle's counterexample with four scalar blocks.
  - Exact value: approx. 0.8723
  - Standard μ upper bound: 1
  - New bound: 0.895

Example: Matrix Copositivity

A matrix $M \in \mathbb{R}^{n \times n}$ is *copositive* if

$$x^T M x \geq 0 \quad \forall x \in \mathbb{R}^n, x_i \geq 0.$$

- The set of copositive matrices is a convex closed cone, but...
- Checking copositivity is coNP-complete
- Very important in QP. Characterization of local solutions.
- The P-satz gives a family of computable SDP conditions, via:

$$(x^T x)^d (x^T M x) = s_0 + \sum_i s_i x_i + \sum_{jk} s_{jk} x_j x_k + \cdots$$

Example: Geometric Inequalities

Ono's inequality: For an acute triangle,

$$(4K)^6 \geq 27 \cdot (a^2+b^2-c^2)^2 \cdot (b^2+c^2-a^2)^2 \cdot (c^2+a^2-b^2)^2$$

where K and a, b, c are the area and lengths of the edges.
The inequality is true if:

$$\left.\begin{aligned} t_1 := a^2+b^2-c^2 &\geq 0 \\ t_2 := b^2+c^2-a^2 &\geq 0 \\ t_3 := c^2+a^2-b^2 &\geq 0 \end{aligned}\right\} \Rightarrow (4K)^6 \geq 27 \cdot t_1^2 \cdot t_2^2 \cdot t_3^2$$

A simple proof: define

$$s(x,y,z) = (x^4+x^2y^2-2y^4-2x^2z^2+y^2z^2+z^4)^2 + 15 \cdot (x-z)^2(x+z)^2(z^2+x^2-y^2)^2.$$

We have then

$$(4K)^6 - 27 \cdot t_1^2 \cdot t_2^2 \cdot t_3^2 = s(a,b,c) \cdot t_1 \cdot t_2 + s(c,a,b) \cdot t_1 \cdot t_3 + s(b,c,a) \cdot t_2 \cdot t_3$$

therefore *proving* the inequality.