Optimal Transceiver Design for Multi-Access Communication

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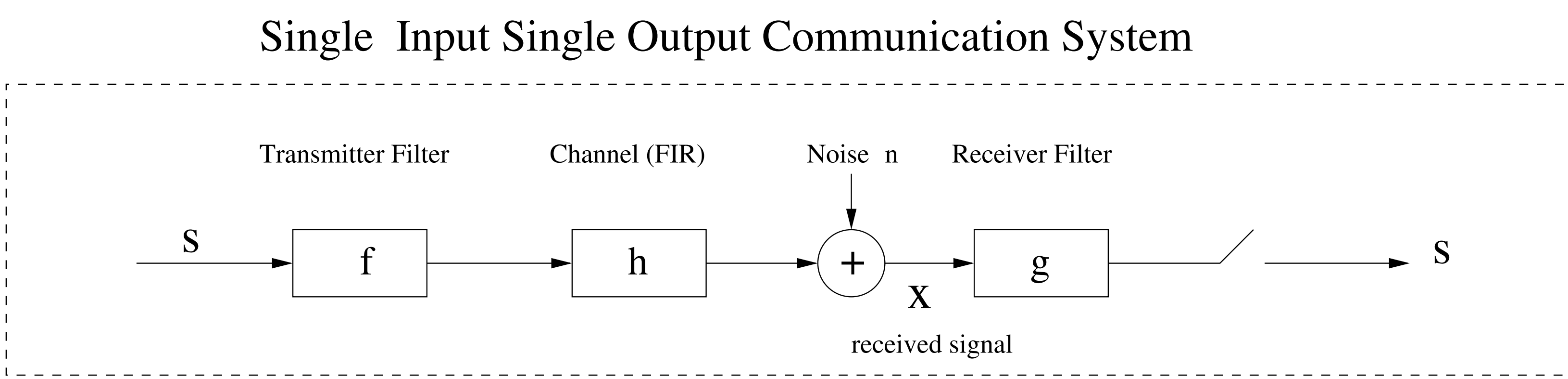
Main Points

- An important problem in the management of communication networks: *resource allocation*
- Frequency, transmitting power; Goal: high data rate, low bit error rate
- Transmitter + receiver for multi-user high speed broadband Digital Subscribe Line application
- Direct formulation is nonconvex; equivalent formulation is SDP (thus convex);
- Further simplification to SOC, and to combinatorial polynomial time algorithm
- Valuable guidelines and insights for optimal practical transceiver design

Content

- Elements of data communication: OFDM, subcarriers, power loading, precoding/equalization.
- Linear transceiver (Transmitter + Receiver) design for the two user case:
  - SDP formulation
  - SOC formulation
  - $O(n^3)$ strongly polynomial algorithm
- General multi-user case
- Other formulations:
  - Sum-rate transmitter design for multi-access channel
  - Sum-rate transmitter design broadcast channel
  - Linear transceiver design with zero-forcing equalizer

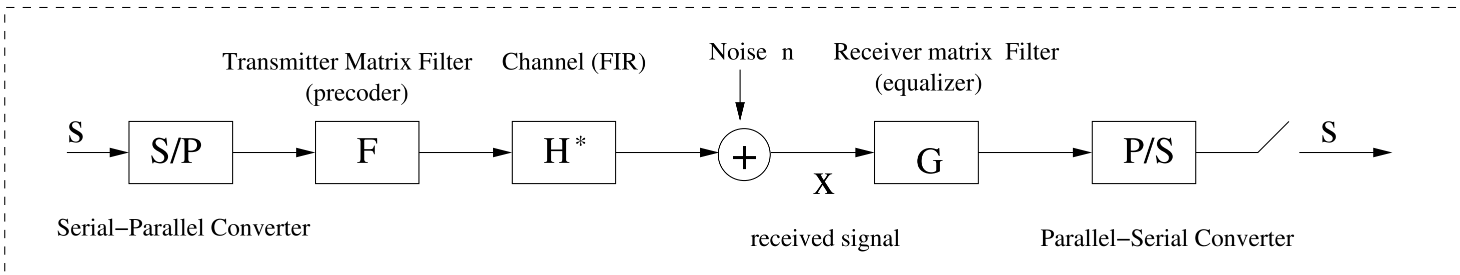
Single User SISO Communication System



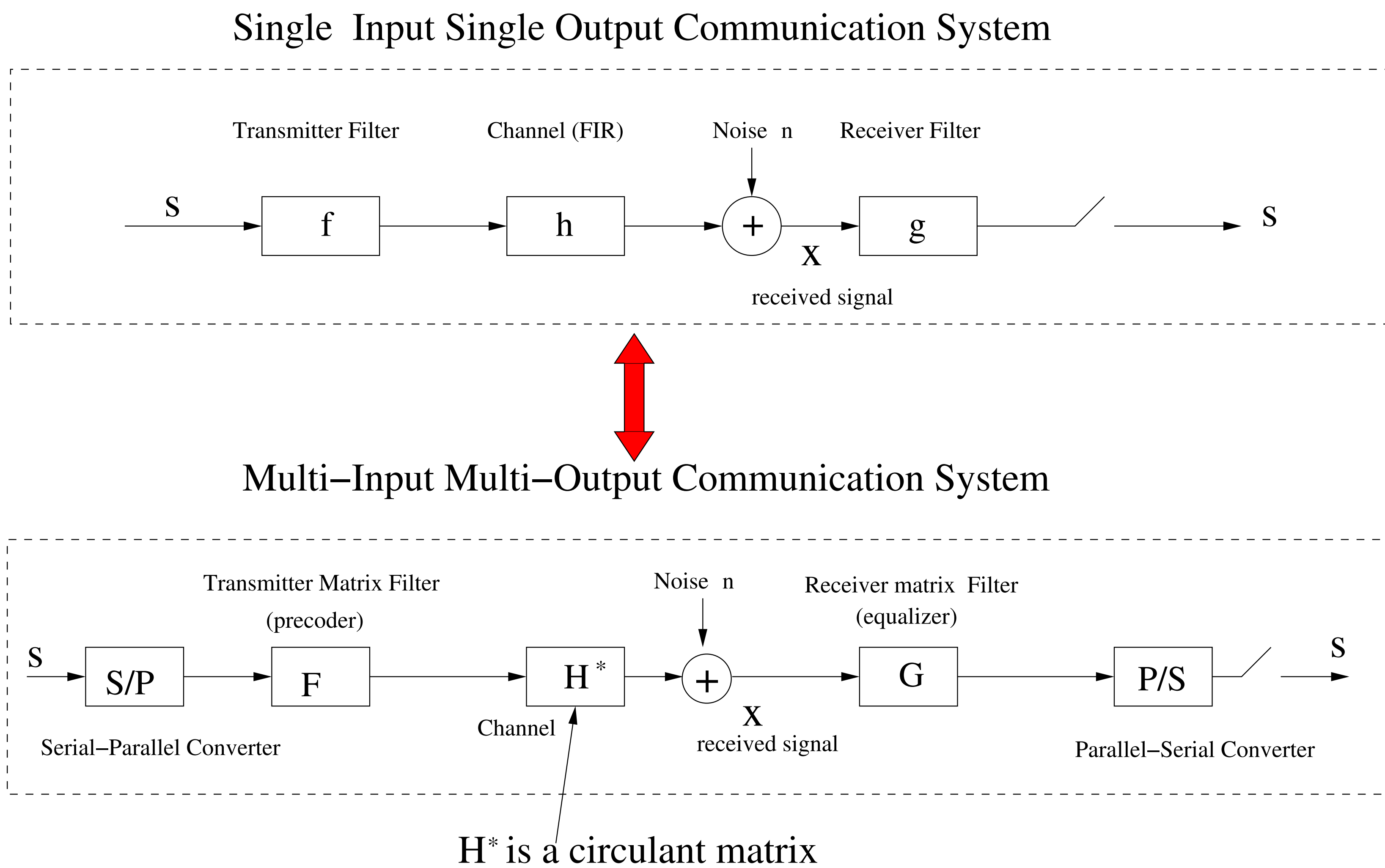
- s is input signal (assumed statistically white)
- $h \in \Re^k$ is a linear, time-invariant channel (assumed known)
- f, g are transmitter filter and equalizer filter respectively
- n is the additive (Gaussian) noise

An Equivalent MIMO System

Multi–Input Multi–Output Communication System

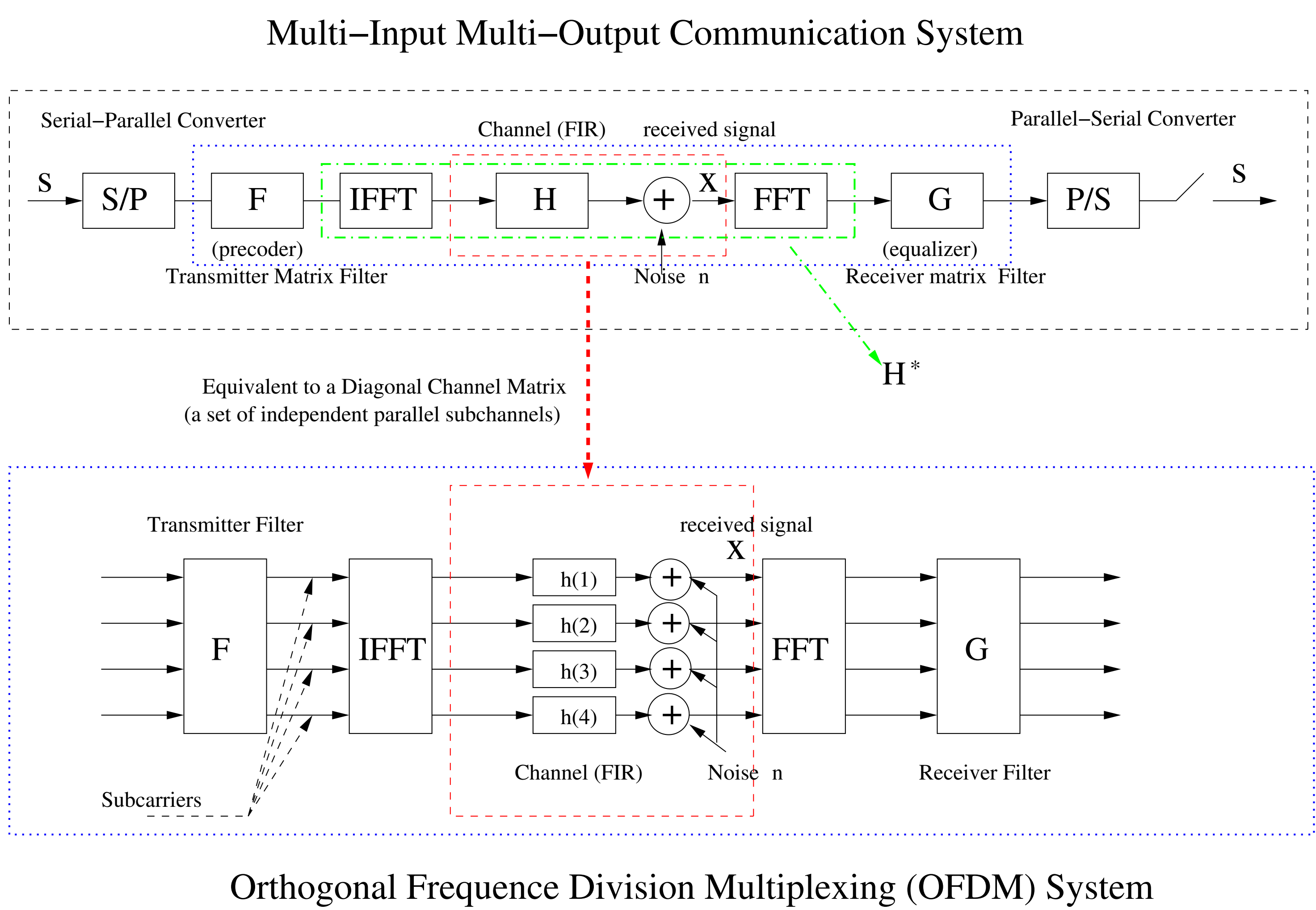


- S/P: serial-parallel converter (with cyclic prefixing); P/S: parallel-serial
- $\mathbf{F}$: $n \times \ell$ transmitter matrix filter (or precoder), obtained from f; data rate $= \ell/n$
- $\mathbf{G}$: $\ell \times n$ receiver equalizer matrix (obtained from g)
- $\mathbf{H}$: channel matrix (obtained from h); $\mathbf{n}$: noise
- $\mathbf{x} = \mathbf{H}^* \mathbf{F}\mathbf{s} + \mathbf{n}$, $\mathbf{H}^*$ is circulant.


Single Input Single Output Communication System
Transmitter Filter
Channel (FIR)
Noise n
Receiver Filter
S
f
h
+
g
X
S
received signal
Multi–Input Multi–Output Communication System
Transmitter Matrix Filter
(precoder)
Noise n
Receiver matrix Filter
(equalizer)
S
S/P
F
H *
+
G
P/S
S
Channel
X
received signal
Serial–Parallel Converter
Parallel–Serial Converter
H* is a circulant matrix

OFDM System

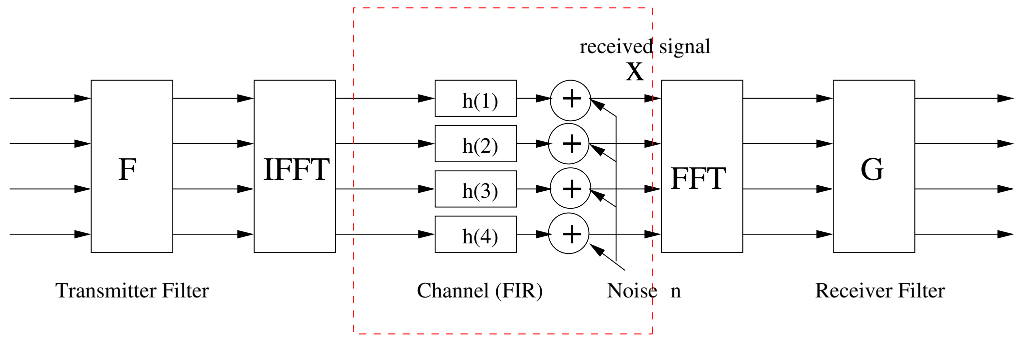
- The circulant channel matrix $\mathbf{H}^*$ can be diagonalized via IFFT/FFT: $\mathbf{H}^* = \mathbf{D}^\dagger \mathbf{H} \mathbf{D}$, where $\mathbf{D}$ is the standard discrete Fourier transform matrix, $\mathbf{H}$ is diagonal
- The diagonalized channel becomes a set of independent subchannels
- Each subchannel corresponds to a subcarrier with a particular frequency (from FFT)
- This diagonalization is not channel dependent
- Orthogonal Frequency Division Multiplexing System employs IFFT/FFT to decompose the channel $\mathbf{H}^*$


Multi−Input Multi−Output Communication System
Serial−Parallel Converter
Channel (FIR)
received signal
Parallel−Serial Converter
S
S/P
F
IFFT
H
+
X
FFT
G
P/S
S
(precoder)
Transmitter Matrix Filter
Noise n
(equalizer)
Receiver matrix Filter
Equivalent to a Diagonal Channel Matrix
(a set of independent parallel subchannels)
H *
Transmitter Filter
received signal
X
F
IFFT
h(1)
h(2)
h(3)
h(4)
FFT
G
Subcarriers
Channel (FIR)
Noise n
Receiver Filter
Orthogonal Frequence Division Multiplexing (OFDM) System

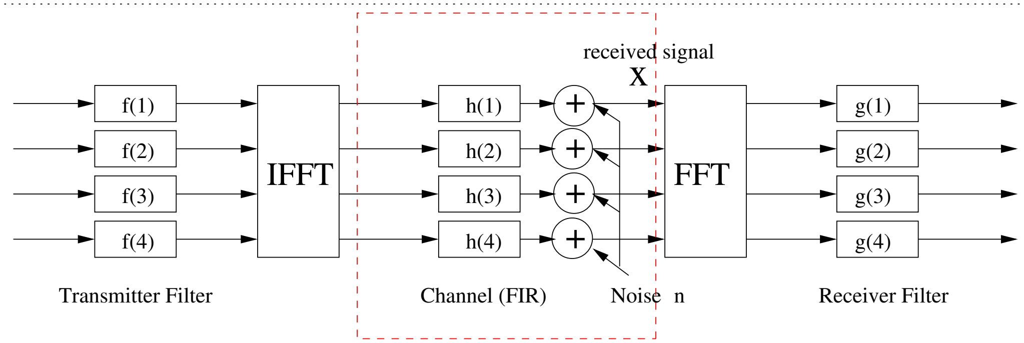
Linearly Precoded/Power Loaded OFDM

- The precoder $\mathbf{F}$ can be a general $n \times n$ matrix, subject to power constraint $\mathrm{tr}(\mathbf{F}\mathbf{F}^{\dagger}) \leq p$.
- The optimized design $\mathbf{F}$ will have a rank $\ell \leq n$, resulting in an optimal data rate ℓ/n.
- A special, and popular, linear precoder is the so called *power loading precoder*: $\mathbf{F}$ is diagonal.

Linearly Precoded/Power Loaded OFDM



General Linearly Precoded OFDM System



Power–Loaded OFDM System

Two-User Multi-Access Communication Channel

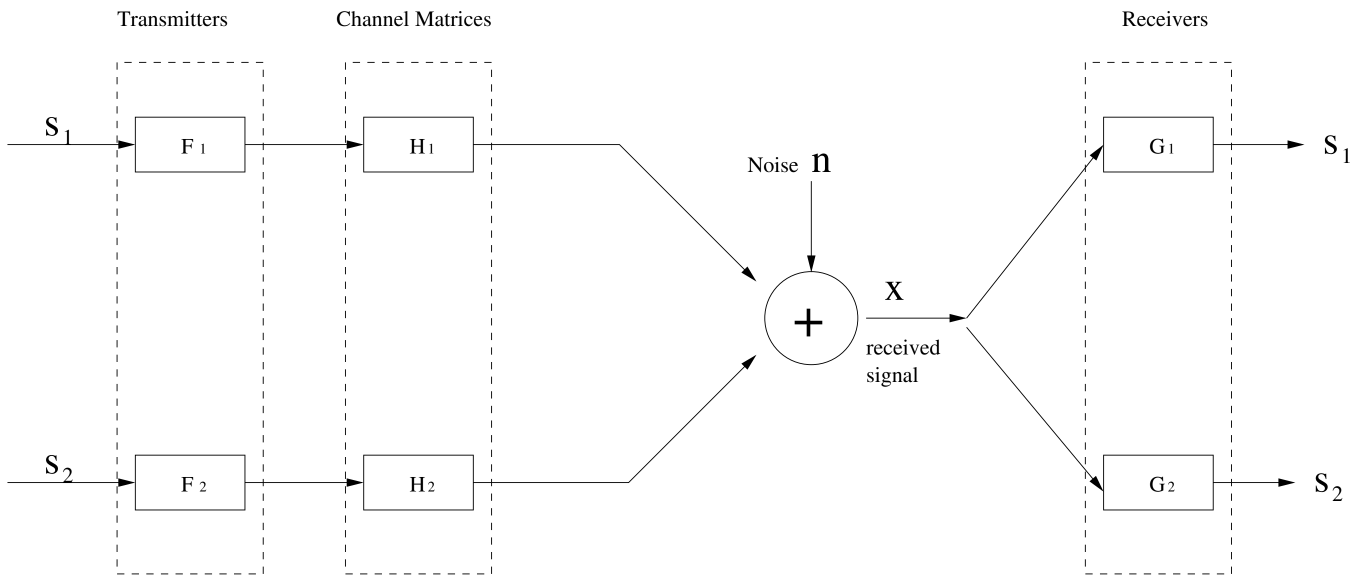


Diagram of Two–user Communication System

Mathematical model: $\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}, \quad \rho > 0.$

Linear detection: $\mathbf{s}_i = \text{sign}\left(\mathbf{G}_i\mathbf{x}\right), \ i = 1, 2.$

Goal: Given the channel matrices, $\mathbf{H}_1$, $\mathbf{H}_2$, design transceivers $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{G}_1$, $\mathbf{G}_2$.

Applications and Previous Work

- Applications include the current and future systems of DSL, DAB, DVB.
- Equalizer (or receiver) design for a fixed transmitter has been researched extensively in the last decade.
- The joint transmitter and receiver (transceiver) design was considered recently, but only for the single user case
- In the single user transceiver design work, the design criteria used include:
  - Minimum Mean Square Error,
  - maximum information rate,
  - channel capacity
- The last two require complex receiver structures.

Mean Square Error

- The detection with receiver (equalizer) $\mathbf{G}_i$: $\hat{\mathbf{s}}_i = \operatorname{sign}\left(\mathbf{G}_i\mathbf{x}\right)$.
- Let $\mathbf{e}_i$ denote the error vector (before making the hard decision) for user i, $i = 1, 2$. Then

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{G}_1\mathbf{x} - \mathbf{s}_1 = \mathbf{G}_1(\mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}) - \mathbf{s}_1 \\ &= \left(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I}\right)\mathbf{s}_1 + \mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{G}_1\mathbf{n}. \end{aligned}$$

- This further implies

$$E(\mathbf{e}_1\mathbf{e}_1^\dagger) = \left(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I}\right)\left(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I}\right)^\dagger + \left(\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\right)\left(\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\right)^\dagger + \rho^2\mathbf{G}_1\mathbf{G}_1^\dagger$$

- Similarly, we have

$$E(\mathbf{e}_2\mathbf{e}_2^\dagger) = \left(\mathbf{G}_2\mathbf{H}_2\mathbf{F}_2 - \mathbf{I}\right)\left(\mathbf{G}_2\mathbf{H}_2\mathbf{F}_2 - \mathbf{I}\right)^\dagger + \left(\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1\right)\left(\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1\right)^\dagger + \rho^2\mathbf{G}_2\mathbf{G}_2^\dagger.$$

Formulation: MMSE Equalizer Case

- Our goal is to design a set of transmitting matrix filters $\mathbf{F}_i$ and a set of matrix equalizers $\mathbf{G}_i$ such that the total mean squared error

$$\mathrm{MSE} = \mathrm{tr}(E(\mathbf{e}_1\mathbf{e}_1^\dagger)) + \mathrm{tr}(E(\mathbf{e}_2\mathbf{e}_2^\dagger))$$

  is minimized.

- As is always the case in practice, there are power constraints on the transmitting matrix filters:

$$\mathrm{tr}(\mathbf{F}_1\mathbf{F}_1^\dagger) \le p_1, \quad \mathrm{tr}(\mathbf{F}_2\mathbf{F}_2^\dagger) \le p_2$$

- The above is nonconvex.

- We first eliminate the variables $\mathbf{G}_1$, $\mathbf{G}_2$: the MMSE equalizers.

Formulation: MMSE Equalizer Case

- By minimizing $E(\mathbf{e}_1\mathbf{e}_1^\dagger)$ with respect to $\mathbf{G}_1$, we obtain the following MMSE equalizer for user 1: $\mathbf{G}_1 = \mathbf{F}_1^\dagger \mathbf{H}_1^\dagger \mathbf{W}$, where

$$\mathbf{W} = \left(\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{F}_2\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger + \rho^2\mathbf{I}\right)^{-1}.$$

- Substituting this into $E(\mathbf{e}_1\mathbf{e}_1^\dagger)$ gives:

$$E(\mathbf{e}_1\mathbf{e}_1^\dagger) = -\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger\mathbf{W}\mathbf{H}_1\mathbf{F}_1 + \mathbf{I}.$$

- Similarly, the MMSE equalizer $\mathbf{G}_2$ for user 2 is given by $\mathbf{G}_2 = \mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\mathbf{W}$ and resulting minimized (with respect to $\mathbf{G}_2$) mean square error for user 2 is given by:

$$E(\mathbf{e}_2\mathbf{e}_2^\dagger) = -\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\mathbf{W}\mathbf{H}_2\mathbf{F}_2 + \mathbf{I}.$$

Total MSE

Substituting into the above expression gives rise to

$$\begin{aligned}
\mathrm{MSE} &= \mathrm{tr}(E(\mathbf{e}_1\mathbf{e}_1^\dagger)) + \mathrm{tr}(E(\mathbf{e}_2\mathbf{e}_2^\dagger)) \\
&= -\mathrm{tr}\left(\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger\mathbf{W}\mathbf{H}_1\mathbf{F}_1\right) - \mathrm{tr}\left(\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\mathbf{W}\mathbf{H}_2\mathbf{F}_2\right) + 2n \\
&= -\mathrm{tr}\left(\mathbf{W}\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger\right) - \mathrm{tr}\left(\mathbf{W}\mathbf{H}_2\mathbf{F}_2\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\right) + 2n \\
&= -\mathrm{tr}\left(\mathbf{W}(\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{F}_2\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger)\right) + 2n \\
&= \rho^2\mathrm{tr}\left(\mathbf{W}\right) + n,
\end{aligned}$$

where the last step follows from the definition of $\mathbf{W}$.

Formulation: MMSE Equalizer Case

- Introduce matrix variables: $\mathbf{U}_1 = \mathbf{F}_1\mathbf{F}_1^\dagger, \quad \mathbf{U}_2 = \mathbf{F}_2\mathbf{F}_2^\dagger.$
- Then the MMSE transceiver design problem becomes

$$\begin{array}{ll} \text{minimize}_{\mathbf{U}_1,\mathbf{U}_2} & \operatorname{tr}\left((\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I})^{-1} \right) \\ \text{subject to} & \operatorname{tr}(\mathbf{U}_1) \le p_1, \quad \operatorname{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{U}_1 \succeq \mathbf{0}, \quad \mathbf{U}_2 \succeq \mathbf{0}. \end{array}$$

- Reformulate using the auxiliary matrix variable $\mathbf{W}$:

$$\begin{array}{ll} \text{minimize}_{\mathbf{W},\mathbf{U}_1,\mathbf{U}_2} & \operatorname{tr}(\mathbf{W}) \\ \text{subject to} & \operatorname{tr}(\mathbf{U}_1) \le p_1, \quad \operatorname{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{W} \succeq (\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I})^{-1} \\ & \mathbf{U}_1 \succeq \mathbf{0}, \quad \mathbf{U}_2 \succeq \mathbf{0}. \end{array}$$

SDP Formulation

- The constraint $\mathbf{W} \succeq (\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I})^{-1}$ is equivalent to LMI:

$$\begin{bmatrix} \mathbf{W} & \mathbf{I} \\ \mathbf{I} & \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I} \end{bmatrix} \succeq 0. \qquad (3)$$

- We obtain an SDP formulation:

$$\begin{array}{ll} \text{minimize}_{\mathbf{W},\mathbf{U}_1,\mathbf{U}_2} & \mathrm{tr}\,(\mathbf{W}) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{W} \text{ satisfies (3)}, \\ & \mathbf{U}_1 \succeq \mathbf{0}, \quad \mathbf{U}_2 \succeq \mathbf{0}. \end{array}$$

- Interior point method with arithmetic complexity $O(n^{6.5}\log(1/\epsilon))$, $\epsilon > 0$ is the solution accuracy.

OFDM: Diagonal Designs are Optimal!

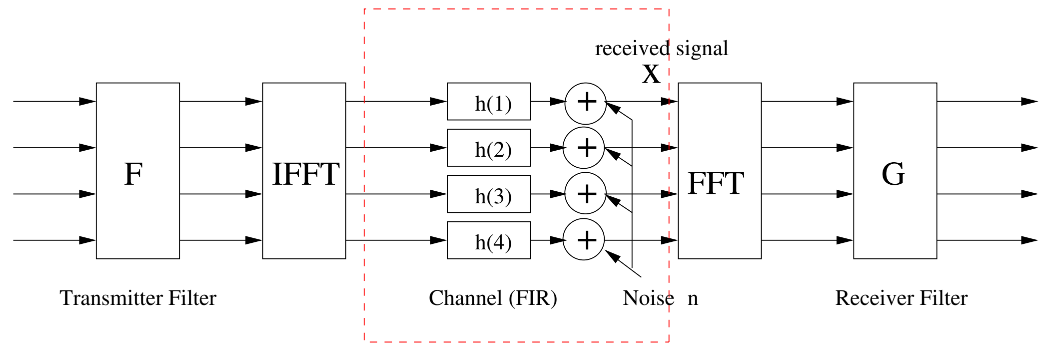
Result

If $\mathbf{H}_1$ and $\mathbf{H}_2$ are diagonal, as in the OFDM systems, then the optimal transmitters are also diagonal.

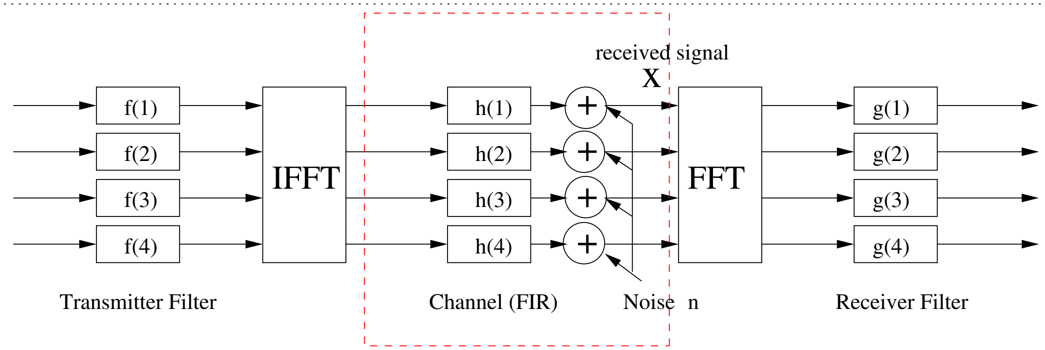
Implication

The MMSE transceivers for an multi-user OFDM system can be implemented by optimally setting the data rates and allocating power to each subcarrier for all the users.

Linearly Precoded/Power Loaded OFDM



General Linearly Precoded OFDM System



Power–Loaded OFDM System

From SDP to SOC Formulation

- Restricting to diagonal designs, the SDP becomes SOC:

$$
\begin{array}{ll}
\text{minimize}_{\mathbf{w},\mathbf{u}_1,\mathbf{u}_2} & \sum_{i=1}^{n} \mathbf{w}_i \\
\text{subject to} & \sum_{i=1}^{n} \mathbf{u}_1(i) \le p_1, \quad \sum_{i=1}^{n} \mathbf{u}_2(i) \le p_2, \\
& \mathbf{w}_i \left(|\mathbf{h}_1(i)|^2 \mathbf{u}_1(i) + |\mathbf{h}_2(i)|^2 \mathbf{u}_2(i) + \rho^2 \right) \ge 1, \\
& \mathbf{u}_1(i) \ge 0, \quad \mathbf{u}_2(i) \ge 0, \quad i = 1, 2, \ldots, n.
\end{array}
$$

- There exist highly efficient (general purpose) interior point methods to solve the above second order cone program.
- Arithmetic complexity $O(n^{3.5} \log(1/\epsilon))$, $\epsilon > 0$ is the accuracy.

Properties of Optimal MMSE Transceiver

- Let $\mathbf{u}_1^* \geq 0$, $\mathbf{u}_2^* \geq 0$ be the optimal transceivers. Define:

$$\begin{cases} I_1 = \{i \mid \mathbf{u}_1^*(i) > 0,\ \mathbf{u}_2^*(i) = 0\}, & I_2 = \{i \mid \mathbf{u}_1^*(i) = 0,\ \mathbf{u}_2^*(i) > 0\}, \\ I_s = \{i \mid \mathbf{u}_1^*(i) > 0,\ \mathbf{u}_2^*(i) > 0\}, & I_u = \{i \mid \mathbf{u}_1^*(i) = 0,\ \mathbf{u}_2^*(i) = 0\}. \end{cases}$$

- I_1, I_2: subcarriers allocated to user 1 and user 2;
  I_s and I_u: subcarriers *shared* and *unused*;
  data rates: $(|I_1| + |I_s|)/n$, $(|I_2| + |I_s|)/n$

- – For each $i \in I_1$ and $j \in I_2$, we have $\frac{|\mathbf{h}_1(i)|^2}{|\mathbf{h}_2(i)|^2} \geq \frac{|\mathbf{h}_1(j)|^2}{|\mathbf{h}_2(j)|^2}$.
  – For all $i, j \in I_s$, we have $\frac{|\mathbf{h}_1(i)|^2}{|\mathbf{h}_2(i)|^2} = \frac{|\mathbf{h}_1(j)|^2}{|\mathbf{h}_2(j)|^2}$.
  – For any $i \in I_u$ and any $j \in I_1 \cup I_s$, we have $|\mathbf{h}_1(i)|^2 < |\mathbf{h}_1(j)|^2$. Similarly, for any $i \in I_u$ and any $j \in I_2 \cup I_s$, we have $|\mathbf{h}_2(i)|^2 < |\mathbf{h}_2(j)|^2$.

Intuitive Interpretation

- $\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}$, with $\mathbf{H}_i$, $\mathbf{F}_i$ diagonal; $\mathbf{x}(i) = \mathbf{h}_1(i)\mathbf{f}_1(i)\mathbf{s}_1(i) + \mathbf{h}_2(i)\mathbf{f}_2(i)\mathbf{s}_2(i) + \rho^2\mathbf{n}(i)$.
- In a fading environment, the path gains $|\mathbf{h}_1(i)|^2$, $|\mathbf{h}_2(i)|^2$ are random,

  $\Rightarrow$ the probability of having two equal path gains is zero.

  $\Rightarrow$ I_s is singleton: *at most one subcarrier should be shared by the two users.*
- The remaining subcarriers are allocated to the two users according to the path gain ratios: subcarrier i to user 1 and subcarrier j to user 2 only if

$$\frac{|\mathbf{h}_1(i)|^2}{|\mathbf{h}_2(i)|^2} \geq \frac{|\mathbf{h}_1(j)|^2}{|\mathbf{h}_2(j)|^2}.$$

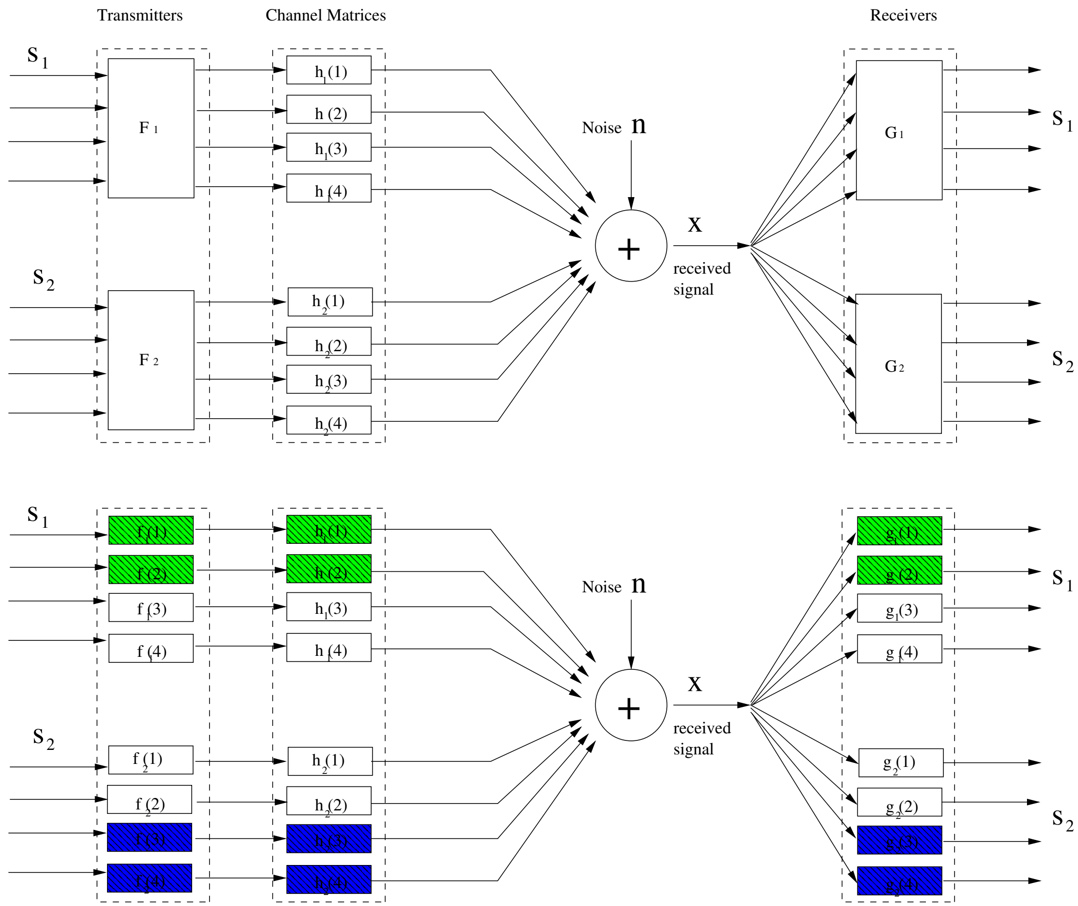
- The subcarriers in I_u have small path gains for both users (i.e., both $|\mathbf{h}_1(i)|^2$ and $|\mathbf{h}_2(i)|^2$ are small), and they should not be used by either user, i.e., they are useless subcarriers!

A Strongly Polynomial Time Algorithm

- The properties of optimal MMSE transceivers can be used to design a *combinatorial* algorithm.
- Assume
$$\frac{|\mathbf{h}_1(1)|^2}{|\mathbf{h}_2(1)|^2} > \frac{|\mathbf{h}_1(2)|^2}{|\mathbf{h}_2(2)|^2} > \cdots > \frac{|\mathbf{h}_1(n-1)|^2}{|\mathbf{h}_2(n-1)|^2} > \frac{|\mathbf{h}_1(n)|^2}{|\mathbf{h}_2(n)|^2}.$$
- Then $I_1 \subseteq \{1, ..., i\}$ and $I_2 \subseteq \{i, ..., n\}$ for some i.
- Leads to an $O(n^3)$ strongly polynomial time (combinatorial) algorithm (vs. $O(n^{3.5} \log 1/\epsilon)$ interior point algorithm for SOC).

Practical Implications

Subcarrier Allocation and Power Loading



General m-User Case

- Mathematical model:

$$\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \cdots + \mathbf{H}_m\mathbf{F}_m\mathbf{s}_m + \rho\mathbf{n}.$$

- Let $\mathbf{G}_i$ be the linear MMSE matrix equalizer at the i-th receiver. Then the total MSE is given by

$$\rho^2 \mathrm{tr}\left((\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger + \cdots + \mathbf{H}_i\mathbf{F}_i\mathbf{F}_i^\dagger\mathbf{H}_i^\dagger + \rho^2\mathbf{I})^{-1} \right) + (m-1)n.$$

- Let $\mathbf{U}_i = \mathbf{F}_i\mathbf{F}_i^\dagger$. Then the power constrained optimal MMSE transmitter design problem can be described as:

$$\begin{array}{ll} \text{minimize}_{\mathbf{U}_1,\ldots,\mathbf{U}_m} & \mathrm{tr}\left((\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \cdots + \mathbf{H}_m\mathbf{U}_m\mathbf{H}_m^\dagger + \rho^2\mathbf{I})^{-1} \right) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_i) \le p_i, \quad \mathbf{U}_i \succeq \mathbf{0}, \quad i = 1, \ldots, m. \end{array}$$

SDP/SOC Formulation

SDP formulation

$$
\begin{array}{ll}
\text{minimize} & \mathrm{tr}\,(\mathbf{W}) \\
\text{subject to} & \mathrm{tr}(\mathbf{U}_i) \le p_i, \quad \mathbf{U}_i \succeq \mathbf{0}, \quad i = 1, 2, \ldots, m, \\
& \begin{bmatrix} \mathbf{W} & \mathbf{I} \\ \mathbf{I} & \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \cdots + \mathbf{H}_m\mathbf{U}_m\mathbf{H}_m^\dagger + \rho^2\mathbf{I} \end{bmatrix} \succeq 0.
\end{array}
$$

SOC formulation

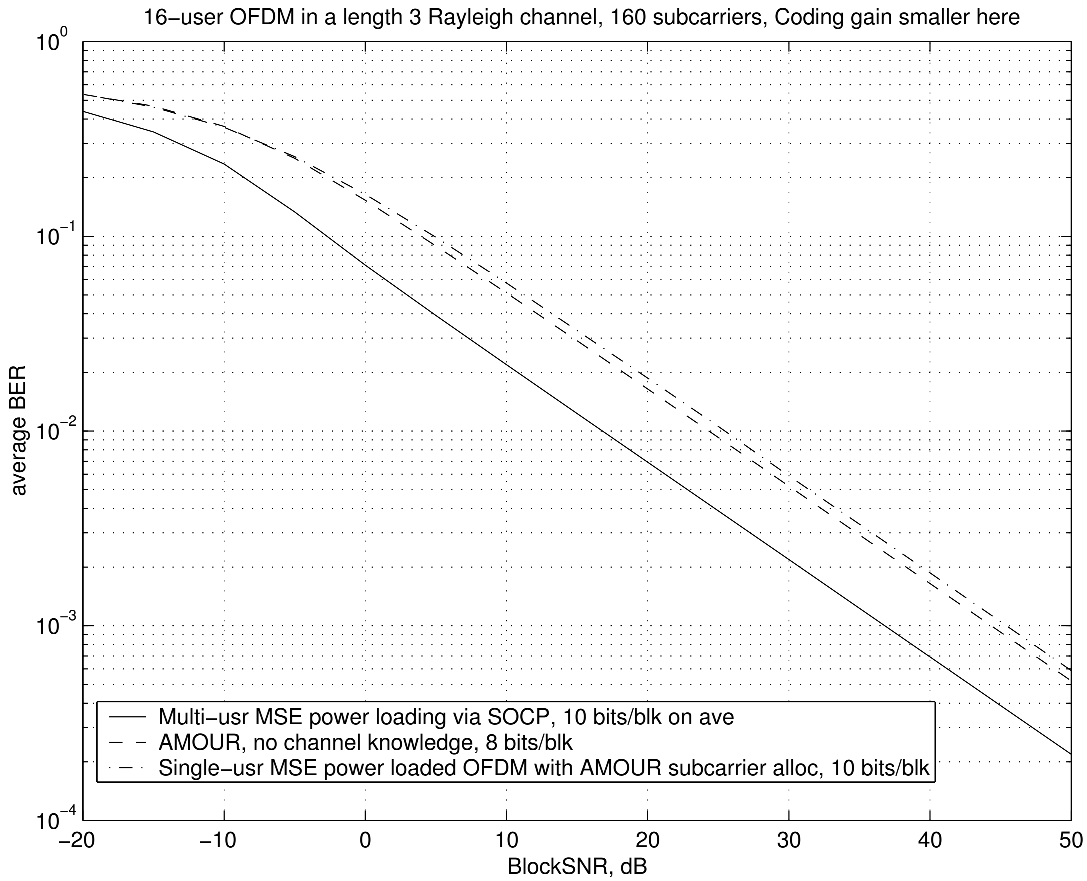
$$
\begin{array}{ll}
\text{minimize}_{\mathbf{w},\mathbf{u}_1,\ldots,\mathbf{u}_m} & \displaystyle\sum_{i=1}^{n} \mathbf{w}_i \\
\text{subject to} & \displaystyle\sum_{i=1}^{n} \mathbf{u}_j(i) \le p_j, \quad j = 1, 2, \ldots, m, \\
& \mathbf{w}_i \left(|\mathbf{h}_1(i)|^2 \mathbf{u}_1(i) + \cdots + |\mathbf{h}_m(i)|^2 \mathbf{u}_m(i) + \rho^2 \right) \ge 1, \\
& \mathbf{u}_j(i) \ge 0, \quad i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m.
\end{array}
$$

Simulation Scenario

- Uplink with 16 active users and 160 available subcarriers
- Each user “sees” its own Rayleigh channel (complex-valued)
- Three schemes:

1. AMOUR – No channel knowledge; Each user uses 10 subcarriers, spreads 8 bits over these carriers using a DFT-type spreading.
2. Individually MMSE power-loaded OFDM – Same subcarrier allocation as AMOUR. Each user sends 1 bit per subcarrier, i.e 10 bits per block; knows its allocated channels and does MMSE power loading for these bits.
3. Multi-user MMSE power loaded OFDM – Using the SOCP formulation. In this case the subcarrier allocations and the number of bits per block vary from block to block, but the average number of bits per block remains 10.

Simulation Results



Efficiency of the Design Approach

On a PIII 600Mhz PC,

- Two users, 2 symbols per block, length 3 channel;
  - SDP $\sim$ 0.65 *secs*
  - SOCP $\sim$ 0.13 *secs*
- 16 users, 10 symbols per block, length 3 channel
  - SOCP $\sim$ 0.65 *secs*

Formulation: Max Sum Rate Capacity for MAC

- Let $\mathbf{\Sigma}_k \succeq 0$, p_k denote the covariance matrix and the transmit power of the k-th user signal.
- The total sum rate of multi-access channel is

$$\log\det(\mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \mathbf{\Sigma}_k \mathbf{H}_k^H)$$

which is achievable by successive nulling and cancellation at BS.

- The multi-user transmitter design is then

$$\begin{array}{ll} \text{maximize} & \log\det(\mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \mathbf{\Sigma}_k \mathbf{H}_k^H) \\ \text{subject to} & \mathrm{tr}(\mathbf{\Sigma}_k) \leq p_k, \quad \mathbf{\Sigma}_k \geq 0, \quad k = 1, 2, \ldots, K. \end{array}$$

- A convex problem; can be solved by interior point methods, iterative water-filling.

Formulation: Max Sum Rate Capacity for BC

- Let $\mathbf{\Sigma}_k \succeq 0$ denote the covariance matrix for the k-th user signal, and let p denote the total transmit power.

- By duality, total sum rate of a broadcast channel channel is

$$\log\det(\mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k^H \bar{\mathbf{\Sigma}}_k \mathbf{H}_k)$$

  which is achievable by dirty paper coding technique, where $\bar{\mathbf{\Sigma}}_k$ (new variables) depends on $\mathbf{\Sigma}_k$ linearly.

- The multi-user transmitter design is then

$$\begin{array}{ll} \text{maximize} & \log\det(\mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k \bar{\mathbf{\Sigma}}_k \mathbf{H}_k^H) \\ \text{subject to} & \sum_{k=1}^{K} \mathrm{tr}(\bar{\mathbf{\Sigma}}_k) \le p, \quad \bar{\mathbf{\Sigma}}_k \ge 0, \quad k = 1, 2, \ldots, K. \end{array}$$

- A convex problem; can be solved by interior point methods (and iterative water-filling?).

Formulation: Zero-Forcing Equalizer Case

- Recall $\mathbf{e}_i$ denotes the error vector (before making the hard decision) for user i, $i = 1, 2$ and

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{G}_1\mathbf{x} - \mathbf{s}_1 = \mathbf{G}_1(\mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}) - \mathbf{s}_1 \\ &= (\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I})\,\mathbf{s}_1 + \mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{G}_1\mathbf{n}. \end{aligned}$$

- Moreover,

$$E(\mathbf{e}_1\mathbf{e}_1^\dagger) = (\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I})\,(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I})^\dagger + (\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2)\,(\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2)^\dagger + \rho^2\mathbf{G}_1\mathbf{G}_1^\dagger$$

$$E(\mathbf{e}_2\mathbf{e}_2^\dagger) = (\mathbf{G}_2\mathbf{H}_2\mathbf{F}_2 - \mathbf{I})\,(\mathbf{G}_2\mathbf{H}_2\mathbf{F}_2 - \mathbf{I})^\dagger + (\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1)\,(\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1)^\dagger + \rho^2\mathbf{G}_2\mathbf{G}_2^\dagger.$$

- Consider the zero-forcing equalizers:

$$\mathbf{G}_1 = (\mathbf{H}_1\mathbf{F}_1)^{-1}, \quad \mathbf{G}_2 = (\mathbf{H}_2\mathbf{F}_2)^{-1}.$$

Formulation: Zero-Forcing Equalizer Case

- Substituting the ZF conditions into the MSE expressions gives

$$\begin{aligned}\mathrm{MSE} &= \mathrm{tr}\left((\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2)(\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2)^\dagger\right) + \rho^2\mathrm{tr}\left(\mathbf{G}_1\mathbf{G}_1^\dagger\right) \\ &+ \mathrm{tr}\left((\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1)(\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1)^\dagger\right) + \rho^2\mathrm{tr}\left(\mathbf{G}_2\mathbf{G}_2^\dagger\right).\end{aligned}$$

- Introduce new matrix variables

$$\mathbf{U}_1 = \mathbf{F}_1\mathbf{F}_1^\dagger, \quad \mathbf{U}_2 = \mathbf{F}_2\mathbf{F}_2^\dagger, \quad \mathbf{V}_1 = \mathbf{G}_1^\dagger\mathbf{G}_1, \quad \mathbf{V}_2 = \mathbf{G}_2^\dagger\mathbf{G}_2.$$

- Then the MSE can be rewritten as

$$\mathsf{MSE} = \mathrm{tr}(\mathbf{V}_1\mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger) + \rho^2\mathrm{tr}(\mathbf{V}_1) + \mathrm{tr}(\mathbf{V}_2\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger) + \rho^2\mathrm{tr}(\mathbf{V}_2)$$

- The power constraint becomes $\mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2.$
- The ZF condition reduces to $\mathbf{V}_1^{-1} = \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger, \qquad \mathbf{V}_2^{-1} = \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger.$

Formulation: Zero-Forcing Case

- The Minimum MSE transceiver design problem can be cast as

$$\begin{array}{ll} \text{minimize} & \mathrm{MSE} = \mathrm{tr}(\mathbf{V}_1\mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger) + \rho^2\mathrm{tr}(\mathbf{V}_1) + \mathrm{tr}(\mathbf{V}_2\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger) + \rho^2\mathrm{tr}(\mathbf{V}_2) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{V}_1^{-1} = \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger, \quad \mathbf{V}_2^{-1} = \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger, \\ & \mathbf{V}_i \succeq \mathbf{0}, \quad \mathbf{U}_i \succeq \mathbf{0}, \quad i = 1, 2. \end{array}$$

- Note the constraints are nonlinear (due to the matrix inverse)
- The objective function is nonconvex quadratic, due to the cross terms $\mathrm{tr}(\mathbf{V}_1\mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger)$ and $\mathrm{tr}(\mathbf{V}_2\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger)$.
- Reformulation is necessary.

Reformulation: ZF Case

- Use monotonicity and Schur complement technique, we obtain the following equivalent formulation:

$$
\begin{array}{ll}
\text{minimize} & \mathrm{MSE} = \mathrm{tr}(\mathbf{V}_1\mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger) + \rho^2\mathrm{tr}(\mathbf{V}_1) + \mathrm{tr}(\mathbf{V}_2\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger) + \rho^2\mathrm{tr}(\mathbf{V}_2) \\
\text{subject to} & \mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2, \quad \mathbf{V}_i \succeq \mathbf{0},\ \mathbf{U}_i \succeq \mathbf{0}, \quad i = 1,2 \\
& \begin{bmatrix} \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger & \mathbf{I} \\ \mathbf{I} & \mathbf{V}_1 \end{bmatrix} \succeq \mathbf{0}, \qquad \begin{bmatrix} \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger & \mathbf{I} \\ \mathbf{I} & \mathbf{V}_2 \end{bmatrix} \succeq \mathbf{0}.
\end{array} \tag{1}
$$

- Note that the constraints are all linear matrix inequalities (LMIs), and in particular convex.
- But the objective function is nonconvex.

Alternating Direction Method

- Fixing the designs for user 1 (namely, $\mathbf{U}_1$, $\mathbf{V}_1$), the objective function MSE is linear in $\mathbf{U}_2$, $\mathbf{V}_2$, resulting in a SDP. Similarly, fixing $\mathbf{U}_2$ and $\mathbf{V}_2$ yields a semidefinite program in $\mathbf{U}_1$ and $\mathbf{V}_1$.
- **Alternating Direction Method**:
  - At iteration 0, let $\mathbf{U}_i^{(0)} = \mathbf{V}_i^{(0)} = \mathbf{I}$. At iteration $k \geq 1$,
    - * Solve (1) with $\mathbf{U}_2$, $\mathbf{V}_2$ fixed to the values of $\mathbf{U}_2^{(k-1)}$, $\mathbf{V}_2^{(k-1)}$. Update $\mathbf{U}_1^{(k)}$ and $\mathbf{V}_1^{(k)}$ to the resulting optimized values of $\mathbf{U}_1$ and $\mathbf{V}_1$.
    - * Solve (1) with $\mathbf{U}_1$, $\mathbf{V}_1$ fixed to the values of $\mathbf{U}_1^{(k-1)}$, $\mathbf{V}_1^{(k-1)}$. Update $\mathbf{U}_2^{(k)}$ and $\mathbf{V}_2^{(k)}$ to the resulting optimized values of $\mathbf{U}_2$ and $\mathbf{V}_2$.
  - Repeat with $k := k + 1$.
- Convergence: bounded iterates + the minimum principle necessary optimality condition.

Power-loaded OFDM Optimal?

- Let channel matrices $\mathbf{H}_1$, $\mathbf{H}_2$ be diagonal.
- If we fix $\mathbf{U}_2$, $\mathbf{V}_2$ at some positive definite diagonal matrices in (1) and optimize with respect to $\mathbf{U}_1$, $\mathbf{V}_1$, then the resulting optimized matrices $\mathbf{U}_1$, $\mathbf{V}_1$ can also be taken to be positive definite and diagonal.
- The proof uses reduction and a property of bipartite matching polytope.
- **Conjecture**: the optimal solutions of (1) are always diagonal.
- Imply the power-loaded OFDM is optimal.

Diagonal Designs

- Restrict to diagonal designs ($\mathbf{U}_i$, $\mathbf{V}_i$ diagonal)
- The formulation (1) reduces to a geometric program:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \left(\mathbf{v}_1^{-1}(i)\mathbf{v}_2(i) + \mathbf{v}_1(i)\mathbf{v}_2^{-1}(i)\right) + \rho^2 \sum_{i=1}^{n} (\mathbf{v}_1(i) + \mathbf{v}_2(i)) \\ \text{subject to} & \sum_{i=1}^{n} \mathbf{h}_1^{-2}(i)\mathbf{v}_1^{-1}(i) \le p_1, \quad \sum_{i=1}^{n} \mathbf{h}_2^{-2}(i)\mathbf{v}_2^{-1}(i) \le p_2, \\ & \mathbf{v}_j(i) \ge 0, \quad j = 1, 2, \; i = 1, \ldots, n. \end{array} \qquad (2)$$

- (2) can be turned into a convex program by using the standard logarithmic transformation.
- The dual of (2) is a linearly constrained entropy maximization problem.

The Dual Program

$$\begin{array}{ll} \text{maximize} & -\sum_{i=1}^{6n} \delta_i \log \delta_i - \sum_{i=1}^{2n} c_i \delta_{4n+i} + \lambda_1 \log \lambda_1 + \lambda_2 \log \lambda_2 \\ \text{subject to} & \delta_{2i-1} - \delta_{2i} + \rho^2 \delta_{2n+2i-1} - \delta_{4n+i} = 0, \quad 1 \le i \le n, \\ & -\delta_{2i-1} + \delta_{2i} + \rho^2 \delta_{2n+2i} - \delta_{5n+i} = 0, \quad 1 \le i \le n, \\ & \lambda_1 = \sum_{i=4n+1}^{5n} \delta_i, \quad \lambda_2 = \sum_{i=5n+1}^{6n} \delta_i, \\ & \sum_{i=1}^{4n} \delta_i = 1, \quad \delta_i \ge 0, \quad 1 \le i \le 6n, \end{array}$$

where the coefficients c_i are defined as

$$c_i = \begin{cases} \log\left(p_1 \mathbf{h}_1^2(i)\right), & 1 \le i \le n, \\ \log\left(p_2 \mathbf{h}_2^2(i)\right), & n+1 \le i \le 2n. \end{cases}$$

Summary

- So far we have
  - Presented various SDP/SOC formulations and algorithms for the optimal transceiver design problems
  - Studied the properties of the optimal transceiver designs.
  - Demonstrated the potential of SDP/SOC/interior point methods in digital communication.
  - Results provide valuable guidelines and insights for the practical system design.
- Future work
  - Incorporating QoS and other receiver structures in the formulation.
  - Extension to the multi-user downlink case.