Minimum BER Linear Transceivers for Block Communication Systems

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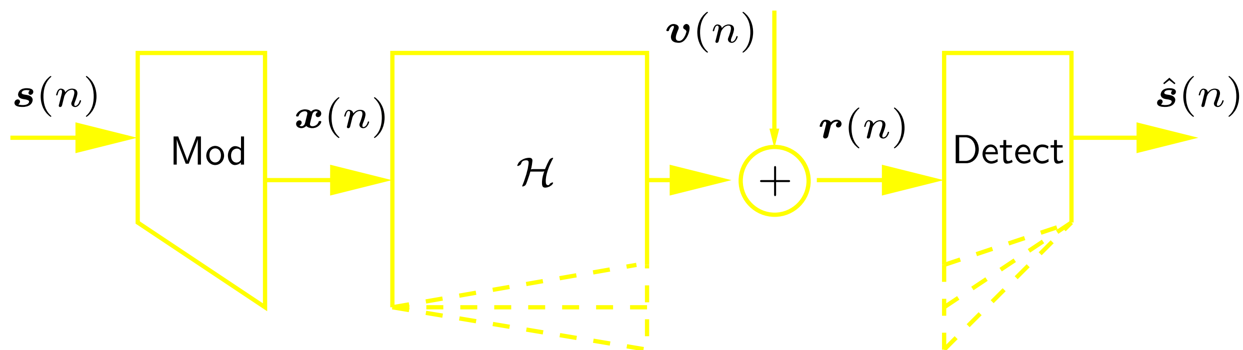
Outline

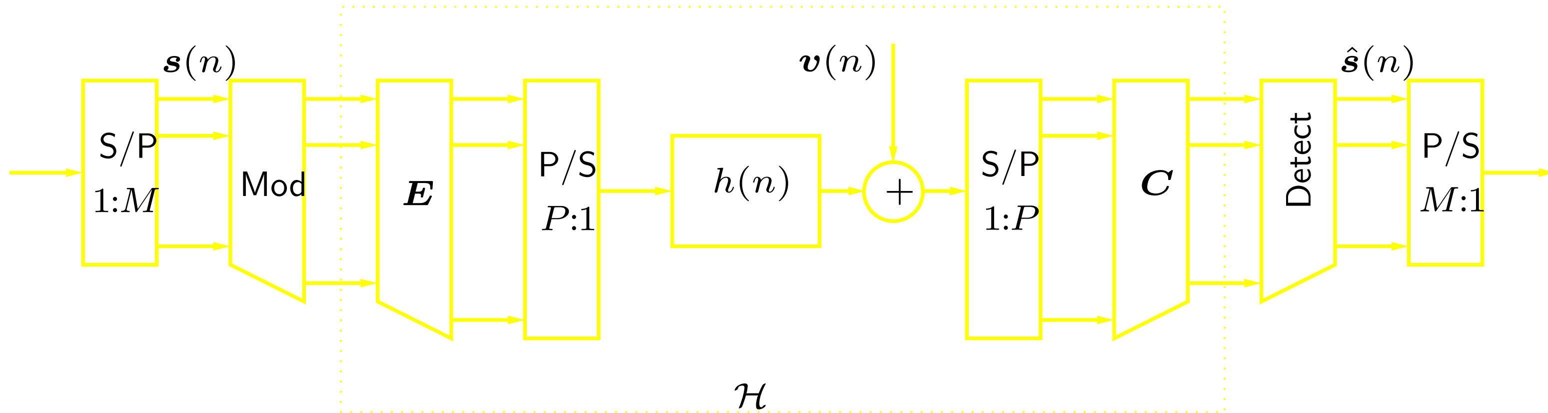
- **Block-by-block communication**
  - **Abstract model**
  - **Applications**
  - **Current design techniques**
- Minimum BER precoders for zero-forcing equalization
  - Average BER and convexity
  - Analytic solution
  - The Message: Minimum BER precoders are MMSE precoders with a special choice of the unitary matrix degree of freedom
  - Performance Analysis: SNR gains of several dB
- Extensions

Block-by-Block Communication



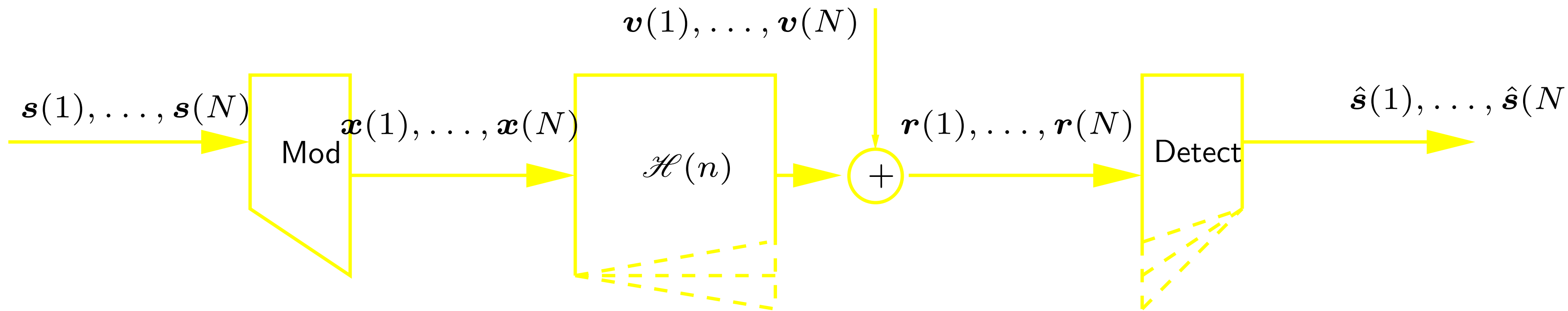
- a vector memoryless system
- $\boldsymbol{r}(n) = \mathcal{H}\boldsymbol{x}(n) + \boldsymbol{v}(n)$
- Applications
  - block transmission over ISI channels (OFDM, DMT)
  - multiple antenna flat fading channels
  - multiple antenna frequency-selective channels

Block transmission over ISI channels



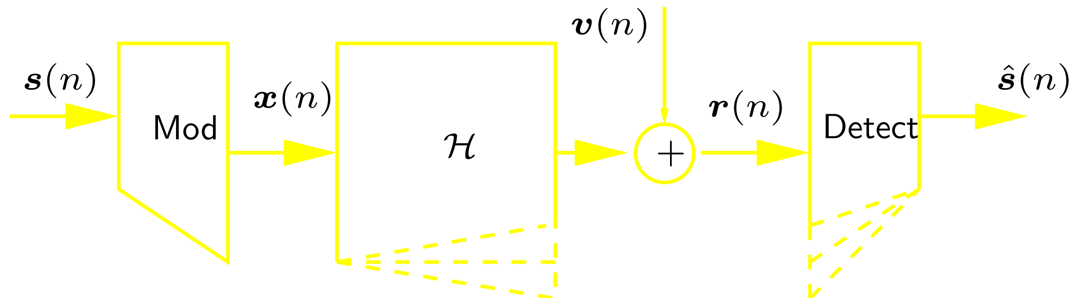
- $\mathcal{H}$ is circulant if $\boldsymbol{E}$ and $\boldsymbol{C}$ are based on cyclic-prefix extensions
- $\mathcal{H}$ is Toeplitz (and full rank) if $\boldsymbol{E}$ and $\boldsymbol{C}$ based on "zero-padding"
- Relative advantages:
  - CP: channel indep. diagonalization via DFT; simple equalization
  - ZP: achieves maximum diversity without coding; no prefix

Multiple antenna flat fading channels



- Flat channels: $\boldsymbol{r}(n) = \mathscr{H}(n)\boldsymbol{x}(n) + \boldsymbol{v}(n)$
- Space-time coding based on accumulating $\boldsymbol{s}(n)$ and $\boldsymbol{r}(n)$ as columns
- If $\mathscr{H}$ is constant, $\boldsymbol{R} = \mathscr{H}\boldsymbol{X} + \boldsymbol{V}$
- Take vec's $\text{vec}(\boldsymbol{R}) = (\boldsymbol{I} \otimes \mathscr{H})\,\text{vec}(\boldsymbol{X}) + \text{vec}(\boldsymbol{V})$
- Hence $\mathcal{H} = \boldsymbol{I} \otimes \mathscr{H}$

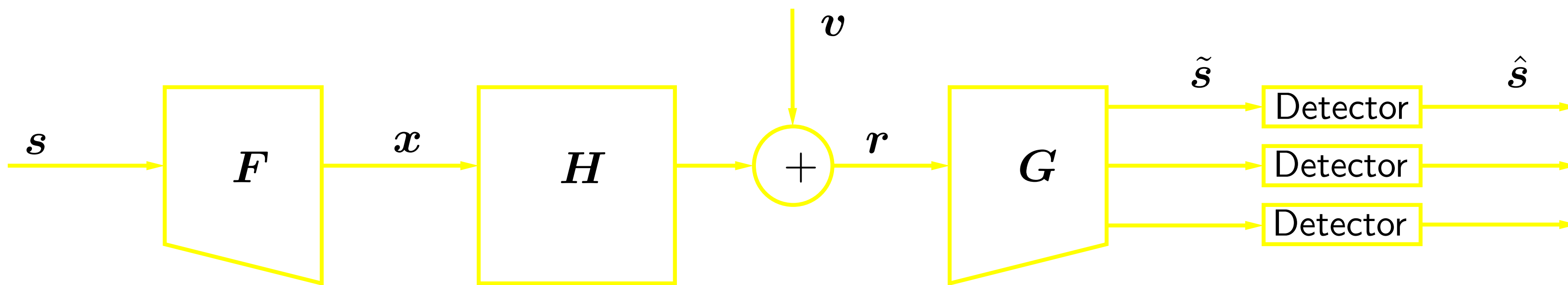
How should we communicate?



Depends on

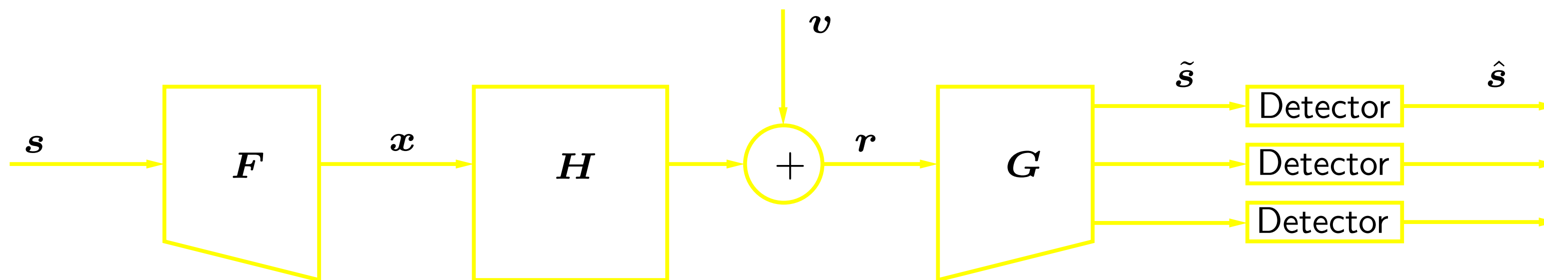
- Structure of the transmitter and receiver
  - in single-user and multiple access transmitter is usually linear, $\boldsymbol{x}(n) = \boldsymbol{F}\boldsymbol{s}(n)$
  - structure of the receiver sometimes fixed
- Accuracy of channel knowledge at the transmitter and receiver
- Design objective
  - rate maximization; performance maximization; power minimization

Today's focus



- Linear transmission
- Receiver: linear equalizer with elementwise detection
- Accurate channel knowledge at both ends
- Performance orientated design
- Uniform constellation assignment in $\boldsymbol{s}$

Current performance orientated designs



Typically, maximize performance subject to power bound for uniform constellation

- Minimize MSE (arithmetic mean of elementwise MSEs)
- Minimize MSE subject to zero forcing equalization
- Maximize total signal power to total noise power subject to zero forcing
- Maximize arithmetic mean of elementwise SINRs
- Maximize geometric mean of SINRs

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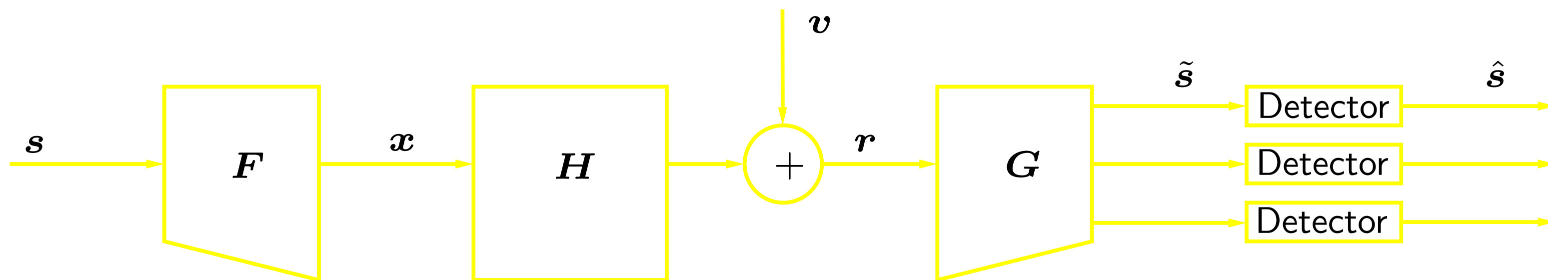
- Block-by-block communication
  - Abstract model
  - Applications
  - Current design techniques
- **Minimum BER precoders for zero-forcing equalization**
  - **Average BER and convexity**
  - **Analytic solution**
  - **The Message: Minimum BER precoders are MMSE precoders with a special choice of the unitary matrix degree of freedom**
  - **Simulation results: SNR gains of several dB**
- Extensions

Analysis



- $\tilde{\boldsymbol{s}} = \boldsymbol{G}\mathcal{H}\boldsymbol{F}\boldsymbol{s} + \boldsymbol{G}\boldsymbol{v}$
- Choose $\mathrm{length}(\boldsymbol{s}) \leq \mathrm{rank}(\mathcal{H})$
- Choose $\boldsymbol{G}$ to be a zero-forcing equalizer: $\boldsymbol{G} = (\mathcal{H}\boldsymbol{F})^{\dagger}$.
- Hence, $\tilde{\boldsymbol{s}} = \boldsymbol{s} + (\mathcal{H}\boldsymbol{F})^{\dagger}\,\boldsymbol{v}$
- Design problem: Find $\boldsymbol{F}$ that minimizes BER

Average bit error rate

- P_e: average bit error rate over all possible transmitted vectors.

$$P_e = E\{P_{e|\boldsymbol{s}}\}$$

- For BPSK/QPSK signals, P_e can be expressed as

$$P_e = \frac{1}{M}\sum_{j=1}^{2^M} P_{\boldsymbol{s}_j} \sum_{m=1}^{M} m P_{m|\boldsymbol{s}_j},$$

- Zero-forcing equalizer $\Longrightarrow$ $\tilde{\boldsymbol{s}} = \boldsymbol{s} + \boldsymbol{G}\boldsymbol{v}$. Hence

$$P_e = \frac{1}{2M}\sum_{m} \mathrm{erfc}\left(\frac{1}{\sqrt{2\sigma^2\left[\boldsymbol{G}\boldsymbol{G}^H\right]_{mm}}}\right)$$

where $E\{\boldsymbol{v}\boldsymbol{v}^H\} = \sigma^2 I$, $[\boldsymbol{G}\boldsymbol{G}^H]_{ii}$ is the ith diagonal entry of $\boldsymbol{G}\boldsymbol{G}^H$, M is the block size.

Key Observation: Convexity

- If $\phi(x) = \mathrm{erfc}\left(\frac{1}{\sqrt{2\sigma^2 x}}\right)$, for $x > 0$, then

$$\frac{d^2\phi}{dx^2} = \frac{1}{\sqrt{\pi}}(2\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2 x}\right)\left(-\frac{3}{2} + \frac{1}{2\sigma^2 x}\right) x^{-\frac{5}{2}}.$$

- Hence, if $x < \frac{1}{3\sigma^2}$, then $\frac{\partial^2 f(x)}{\partial x^2} > 0$.
- Therefore, if

$$\frac{1}{\sigma^2 [\boldsymbol{G}\boldsymbol{G}^H]_{mm}} > 3$$

then P_e is a convex function of $[\boldsymbol{G}\boldsymbol{G}^H]_{mm}$

Design of the Minimum BER Precoder

- Our goal:

$$\begin{array}{ll} \underset{\boldsymbol{F}}{\text{minimize}} & P_e \\ \text{subject to} & \mathrm{trace}(\boldsymbol{F}\boldsymbol{F}^H) \leq p_0 \end{array}$$

- In the region that P_e is convex, applying Jensen's inequality, we have

$$P_e = \frac{1}{2M}\sum_m \mathrm{erfc}\left(\frac{1}{\sqrt{2\sigma^2\,[\boldsymbol{G}\boldsymbol{G}^H]_{mm}}}\right) \geq \frac{1}{2}\mathrm{erfc}\left(\frac{1}{\sqrt{\frac{2\sigma^2}{M}\sum_{m=1}^{M}[\boldsymbol{G}\boldsymbol{G}^H]_{mm}}}\right)$$

$$= \frac{1}{2}\mathrm{erfc}\left(\frac{1}{\sqrt{2\sigma^2(\frac{\mathrm{trace}(\boldsymbol{G}\boldsymbol{G}^H)}{M})}}\right) \triangleq P_{e,LB}$$

equality holds if $[\boldsymbol{G}\boldsymbol{G}^H]_{mm}$ are equal, $\forall m \in [1, M]$.

Design of the Minimum BER Precoder II

- $P_{e,LB}$ defines a lower bound on P_e; minimum BER precoders can be designed in two stages.
  - **Stage 1**: Minimize $P_{e,LB}$ subject to the power constraint and the convex condition.

$$\begin{array}{ll} \underset{\boldsymbol{F}}{\text{minimize}} & P_{e,LB} \\ \text{subject to} & \text{trace}(\boldsymbol{F}\boldsymbol{F}^H) \leq p_0 \\ & [\boldsymbol{G}\boldsymbol{G}^H]_{mm} < \frac{1}{3\sigma^2}, \quad \forall m \in [1, M] \end{array}$$

  - **Stage 2**: Show that a particular solution for **Stage 1** achieves the minimized lower bound.

Solving Stage 1

- Parameterize via SVD: $\boldsymbol{F} = \boldsymbol{U} \left[\begin{smallmatrix} \boldsymbol{\Phi} \\ \boldsymbol{0} \end{smallmatrix} \right] \boldsymbol{V}$; $\quad \boldsymbol{H} = \boldsymbol{Q} \left[\begin{smallmatrix} \boldsymbol{\Sigma} \\ \boldsymbol{0} \end{smallmatrix} \right] \boldsymbol{W}$
- If number columns of $\boldsymbol{F} \leq \text{rank}(\boldsymbol{H})$, then zero forcing equalizers exist
- In those cases $\boldsymbol{G}\boldsymbol{G}^H = \boldsymbol{V}^H \boldsymbol{\Phi}^{-1} \boldsymbol{Z}(\boldsymbol{U}) \boldsymbol{\Phi}^{-1} \boldsymbol{V}$,
  where $\boldsymbol{Z}(\boldsymbol{U}) = \left[\begin{smallmatrix} \boldsymbol{I}_M & \boldsymbol{0} \end{smallmatrix} \right] \boldsymbol{U}^H \boldsymbol{W} \boldsymbol{\Sigma}^{-2} \boldsymbol{W}^H \boldsymbol{U} \left[\begin{smallmatrix} \boldsymbol{I}_M \\ \boldsymbol{0} \end{smallmatrix} \right]$
- Recall that $P_{e,LB} = \frac{1}{2} \text{erfc} \left(\left(2\sigma^2 \, \text{trace}(\boldsymbol{G}\boldsymbol{G}^H)/M \right)^{-1/2} \right)$
- Exploit: monotonicity of $\text{erfc}(\cdot)$;
- Hence, minimizing $P_{e,LB}$ is equivalent to minimizing $\text{trace} \left(\boldsymbol{\Phi}^{-2} \boldsymbol{Z}(\boldsymbol{U}) \right)$

Reformulation

- Minimizing $P_{e,LB}$ subject to power and validity constraints is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{U},\boldsymbol{\Phi},\boldsymbol{V}}{\text{minimize}} & \operatorname{trace}\left(\boldsymbol{\Phi}^{-2}\boldsymbol{Z}(\boldsymbol{U})\right) \\ \text{subject to} & \operatorname{trace}(\boldsymbol{\Phi}^2) \le p_0 \\ & \left[\boldsymbol{V}^H\boldsymbol{\Phi}^{-1}\boldsymbol{Z}(\boldsymbol{U})\boldsymbol{\Phi}^{-1}\boldsymbol{V}\right]_{mm} < \frac{1}{3\sigma^2}, \quad \forall m \in [1, M] \end{array}$$

- Awkward due to the last constraint
- **Lemma:** For symmetric $\boldsymbol{A} \geq \boldsymbol{0}$, with $\boldsymbol{A} = \boldsymbol{\Psi}\boldsymbol{\Gamma}\boldsymbol{\Psi}^H$,

$$\min_{\boldsymbol{V}\boldsymbol{V}^H=\boldsymbol{I}} \max_m [\boldsymbol{V}^H\boldsymbol{A}\boldsymbol{V}]_{mm} = \frac{\operatorname{trace}(\boldsymbol{A})}{M}$$

and a minimizing $\boldsymbol{V} = \boldsymbol{\Psi}\boldsymbol{D}$, where $\boldsymbol{D}$ is the DFT matrix.

Remaining Problem

- To complete minimization of lower bound we must solve:

$$\begin{array}{ll} \underset{\boldsymbol{U},\boldsymbol{\Phi}}{\text{minimize}} & \mathrm{trace}\left(\boldsymbol{\Phi}^{-2}\boldsymbol{Z}(\boldsymbol{U})\right) \\ \text{subject to} & \mathrm{trace}(\boldsymbol{\Phi}^2) \leq p_0 \end{array} \tag{2}$$

- Solution is the MMSE precoder for zero-forcing equalization

$$\boldsymbol{U} = \boldsymbol{W}; \qquad \boldsymbol{\Phi} = \boldsymbol{\Phi}_{\text{MMSE-ZF}} = \sqrt{\frac{p_0}{\mathrm{trace}(\boldsymbol{\Sigma}^{-1})}}\,\boldsymbol{\Sigma}^{-1/2}$$

  where, for simplicity we assumed that $\mathrm{length}(\boldsymbol{s}) = \mathrm{rank}(\boldsymbol{H})$.

- Hence, if $\mathrm{trace}\left(\boldsymbol{\Phi}_{\text{MMSE-ZF}}^{-2}\boldsymbol{Z}(\boldsymbol{W})\right) > M/(3\sigma^2)$ then

$$\boldsymbol{F}_{\text{min, LB}} = \boldsymbol{W}\boldsymbol{\Phi}_{\text{MMSE-ZF}}\boldsymbol{D}$$

  otherwise lower bound is not valid

Minimum BER Precoder

- We now have that $\boldsymbol{F}_{\text{min, LB}} = \boldsymbol{W}\boldsymbol{\Phi}_{\text{MMSE-ZF}}\boldsymbol{D}$
- Does the P_e for this $\boldsymbol{F}$ achieve the lower bound that it minimizes?
- Yes! All $[\boldsymbol{G}\boldsymbol{G}^H]_{mm}$ are equal!
- That is

$$\boldsymbol{F}_{\text{MBER}} = \boldsymbol{W}\boldsymbol{\Phi}_{\text{MMSE-ZF}}\boldsymbol{D}$$

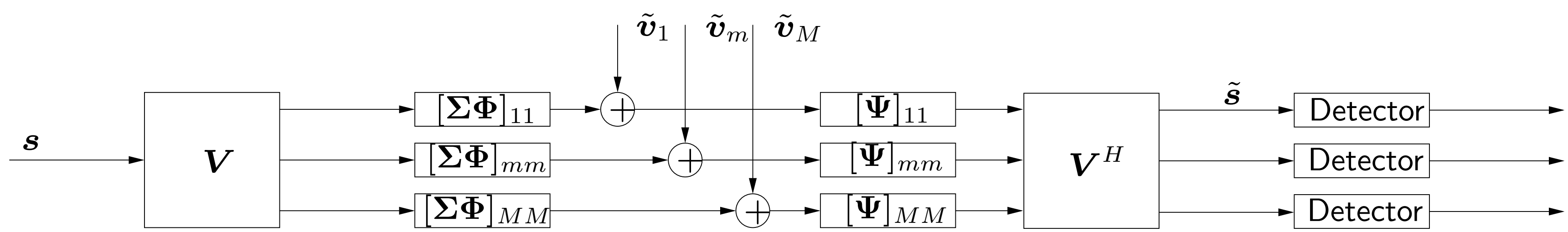
- **Interpretation:**
  - The set of all MMSE precoders for zero-forcing equalization

$$\boldsymbol{F}_{\text{MMSE}} = \boldsymbol{W}\boldsymbol{\Phi}_{\text{MMSE-ZF}}\boldsymbol{V}$$

    for an arbitrary unitary $\boldsymbol{V}$.
  - There are other choices of $\boldsymbol{V}$ which result in minimum BER
  - All that is needed is $\left|[\boldsymbol{V}]_{ij}\right| = 1/\sqrt{M}$

Interpretation II

- For simplicity assume that $\boldsymbol{H}$ and $\boldsymbol{F}$, and hence $\boldsymbol{G}$ are square
- Recall: $\boldsymbol{G} = \boldsymbol{D}^H \boldsymbol{\Psi} \boldsymbol{Q}^H$; $\boldsymbol{H} = \boldsymbol{Q} \boldsymbol{\Sigma} \boldsymbol{W}^H$; $\boldsymbol{F}_{\text{MBER}} = \boldsymbol{W} \boldsymbol{\Phi}_{\text{MMSE-ZF}} \boldsymbol{D}$
- Hence, $\boldsymbol{GHF} = \boldsymbol{V}^H \boldsymbol{\Psi} \boldsymbol{\Sigma} \boldsymbol{\Phi} \boldsymbol{V}$; $\tilde{\boldsymbol{v}} = \boldsymbol{Q}^H \boldsymbol{v}$



- Optimality of structure which synthesizes parallel sub-channels is typical
- However, to achieve minimum BER we must linearly combine these sub-channels

Interpretation III

- Recall: $\boldsymbol{G} = \boldsymbol{D}^H \boldsymbol{\Psi} \boldsymbol{Q}^H$; $\quad \boldsymbol{H} = \boldsymbol{Q}\boldsymbol{\Sigma}\boldsymbol{W}^H$; $\quad \boldsymbol{F}_{\text{MBER}} = \boldsymbol{W}\boldsymbol{\Phi}_{\text{MMSE-ZF}}\boldsymbol{D}$
- $\boldsymbol{\Phi}_{\text{MMSE-ZF}} \propto \boldsymbol{\Sigma}^{-1/2}$; $\quad \boldsymbol{\Psi} = (\boldsymbol{\Sigma}\boldsymbol{\Phi}_{\text{MMSE-ZF}})^{-1} \propto \boldsymbol{\Sigma}^{-1/2}$
- Hence, equalization effort “balanced” between transmitter and receiver
- Transmitter allocates power to compensate for poor sub-channels
- Decision point SNR:

$$\text{SNR}_m = \frac{1}{2\sigma^2[\boldsymbol{G}\boldsymbol{G}^H]_{mm}} \propto \frac{1}{[\boldsymbol{D}\boldsymbol{\Sigma}^{-1}\boldsymbol{D}^H]_{mm}}$$

- Choice of $\boldsymbol{V} = \boldsymbol{D}$ makes all SNR_m equal
- Choice of $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\text{MMSE-ZF}}$ makes this SNR level as small as possible

Sub-channel dropping scheme

- We define block SNR $\rho \triangleq p_0/(P\sigma^2)$.
- Our analytic MBER precoder is valid for moderate-to-high SNR,

$$\rho \geq \frac{3\left(\operatorname{trace}(\mathbf{\Sigma}^{-1})\right)^2}{PM} \triangleq \rho_c,$$

- This condition can be tested before the minimum BER precoder is assembled
- This condition can be ensured by
  - increasing transmitting power
  - dropping the lower-gain sub-channels, reallocating transmission power on the surviving channels.

Minimum BER precoders for CP Systems

- Applying our optimum design $\boldsymbol{F}_{\text{MBER}}$ to the CP system, we have

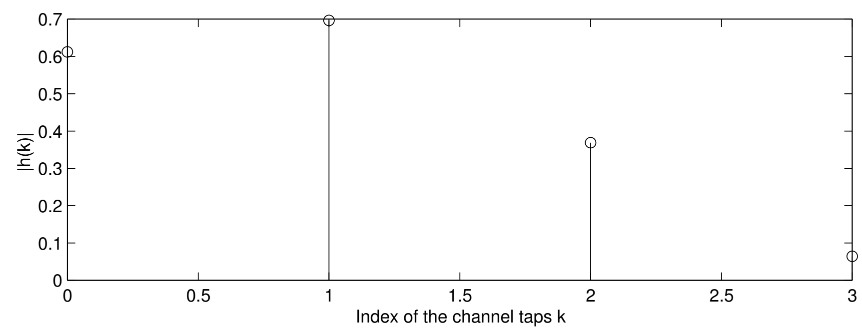
$$\boldsymbol{F}_{\text{CP-MBER}} = \boldsymbol{D}^H \boldsymbol{\Delta}_{\text{MMSE-ZF}} \boldsymbol{D}$$

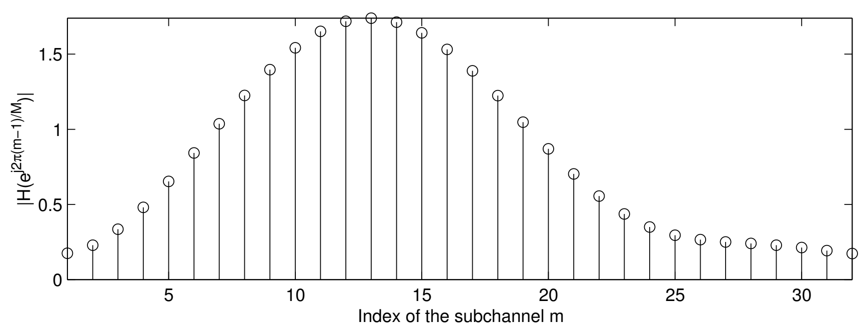
  where $\boldsymbol{\Delta}_{\text{MMSE-ZF}}$ is the MMSE power loading matrix for ZF equalization.

- The CP-MBER precoder is related to standard DMT in that
  - Water-filling power loading is replaced by MMSE power loading
  - the sub-channels are linearly combined via a second DFT matrix

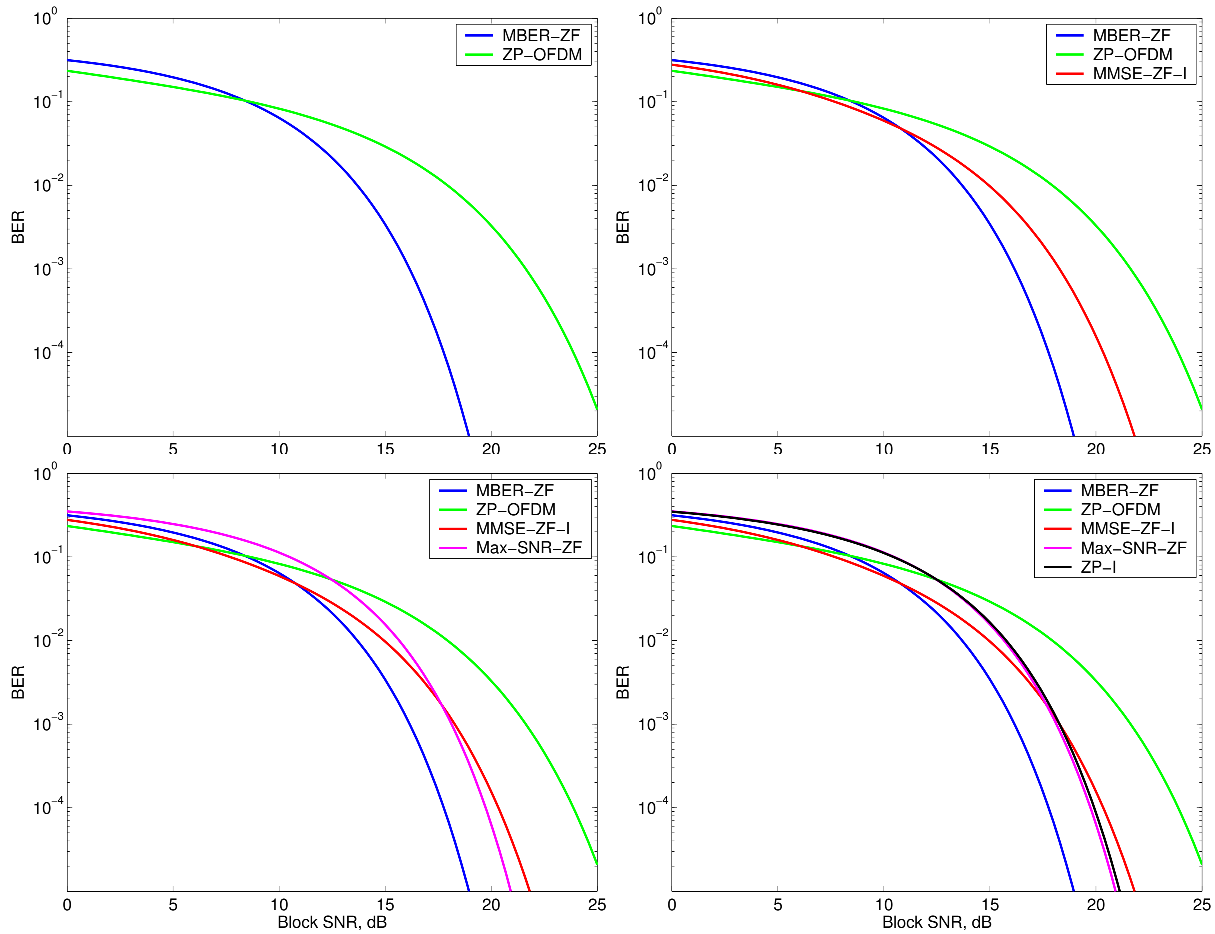
Performance Analysis: One channel

$$(M, L, P) = (32, 3, 35); \qquad \rho = p_0/(P\sigma^2)$$



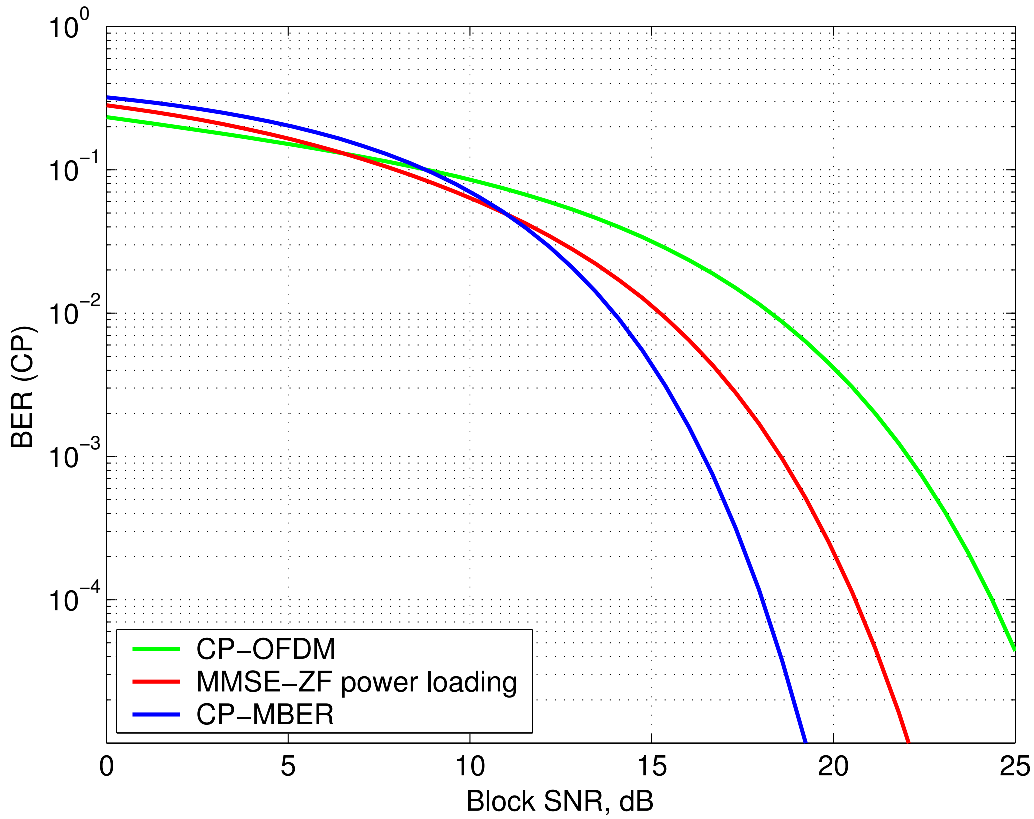


ZP: BER, one channel

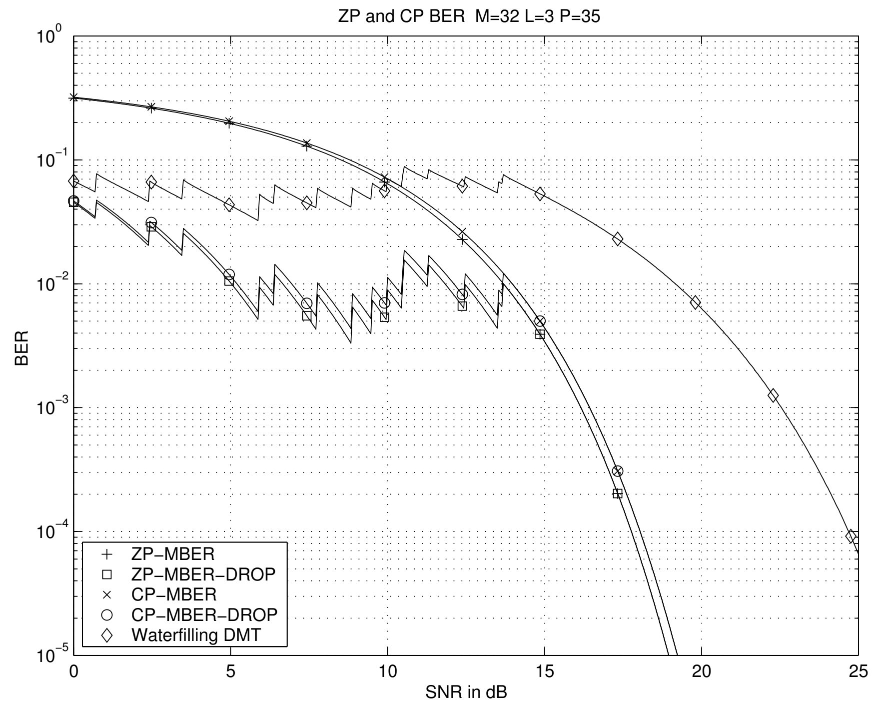


$\boldsymbol{F}_{\text{ZP-OFDM}} \propto \boldsymbol{D}$; SNR gain > 5 dB @ BER$=10^{-4}$

CP: BER, one channel



ZP vs CP, one channel

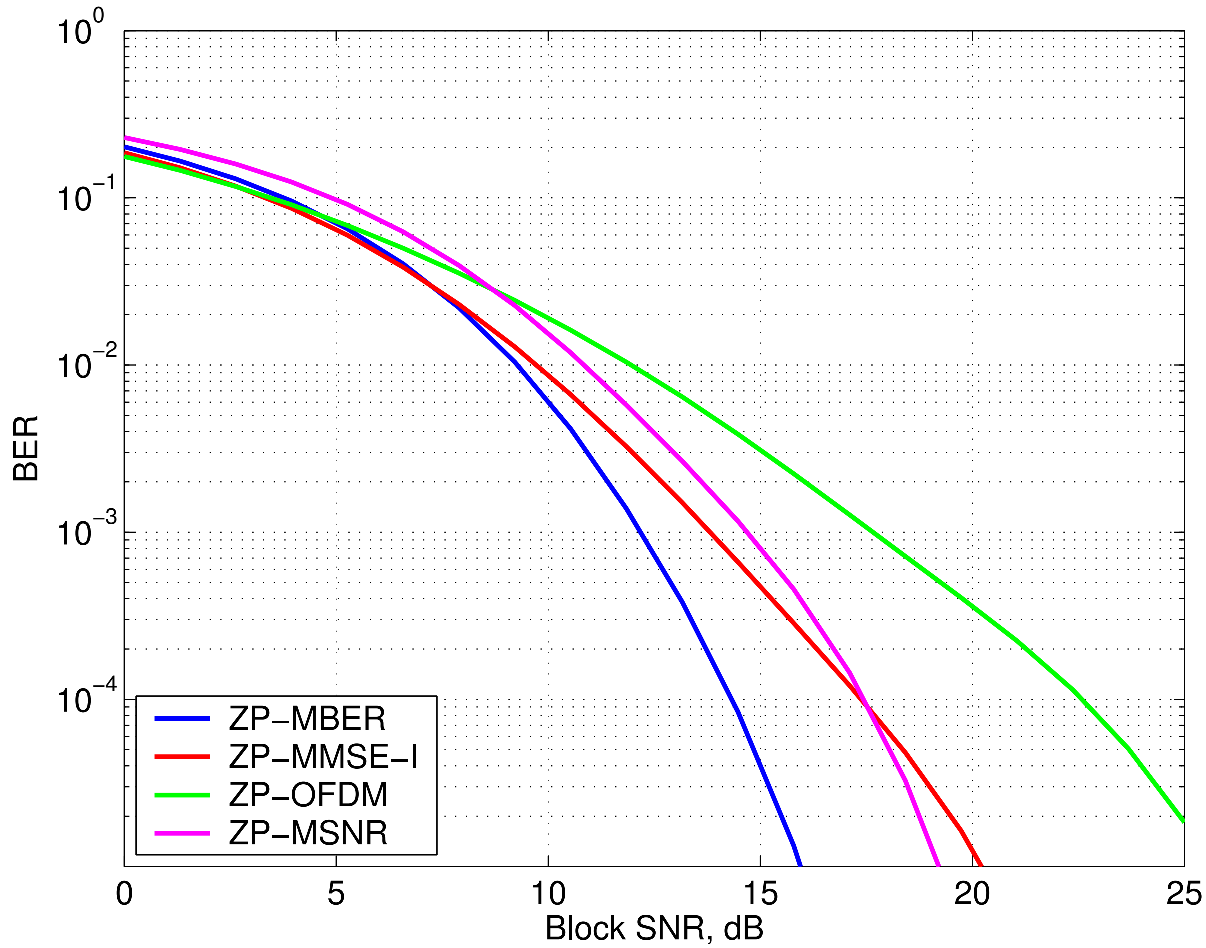


True SNR gain of ZP is larger, as no power wasted sending cyclic prefix

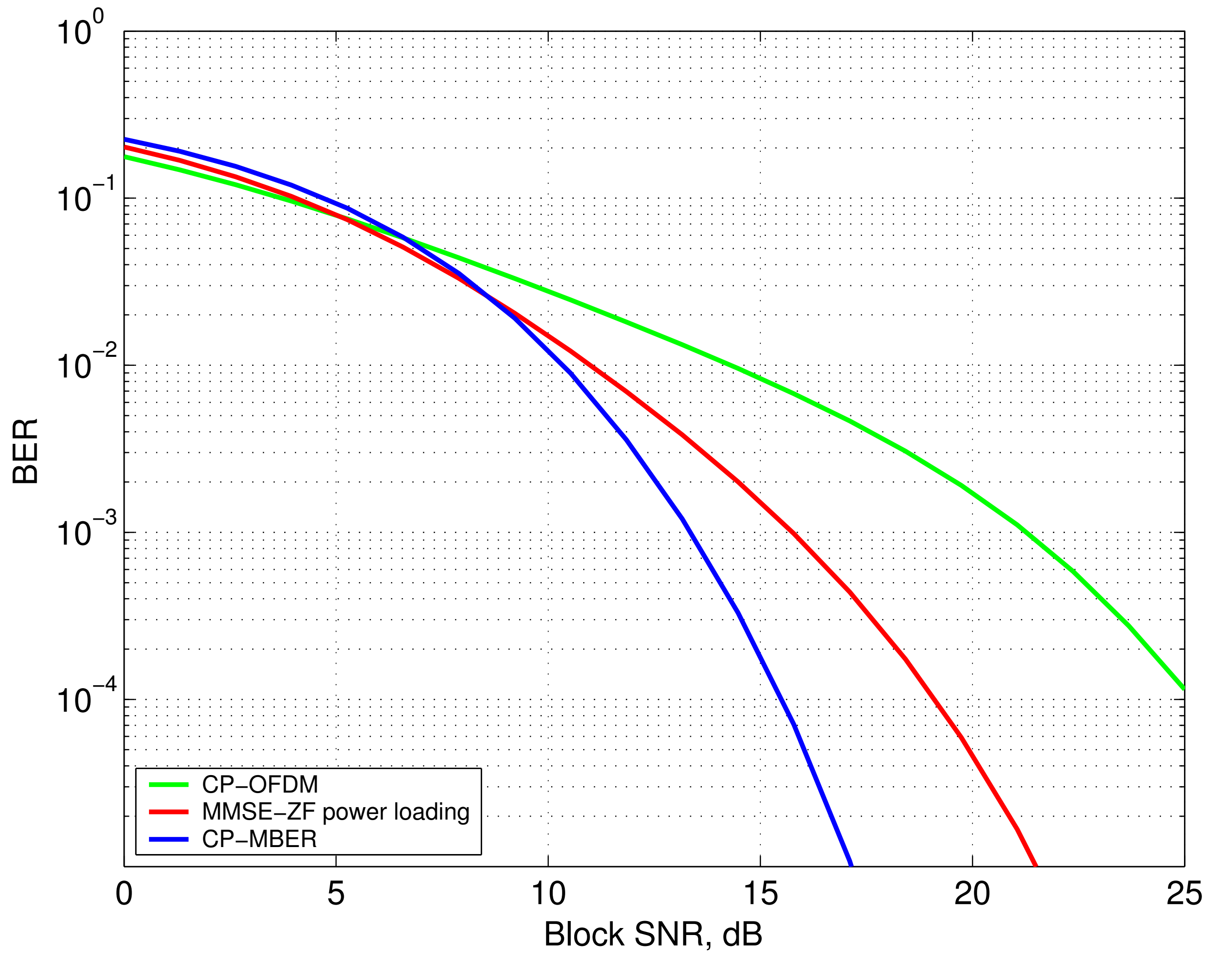
BER analysis: average over channels

- Randomly generated length 5 FIR channels
- i.i.d. zero-mean circular complex Gaussian taps
- impulse response normalized
- $M = 16$, $P = 20$
- Average BER over 2000 realizations

ZP: BER, average over channels



CP: BER, average over channels



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- **Extensions**

Simple extensions

- Higher-order QAM constellations
  - When Gray labelled, BER dominated by nearest and next to nearest neighbour errors
  - Our minimum BER precoder minimizes both simultaneously
- Coloured noise:
  - Current design implicitly depends on eigen structure of $\boldsymbol{H}\boldsymbol{H}^H$
  - Coloured noise case depends on eigen structure of $\boldsymbol{H}\boldsymbol{R}_{vv}^{-1}\boldsymbol{H}^H$

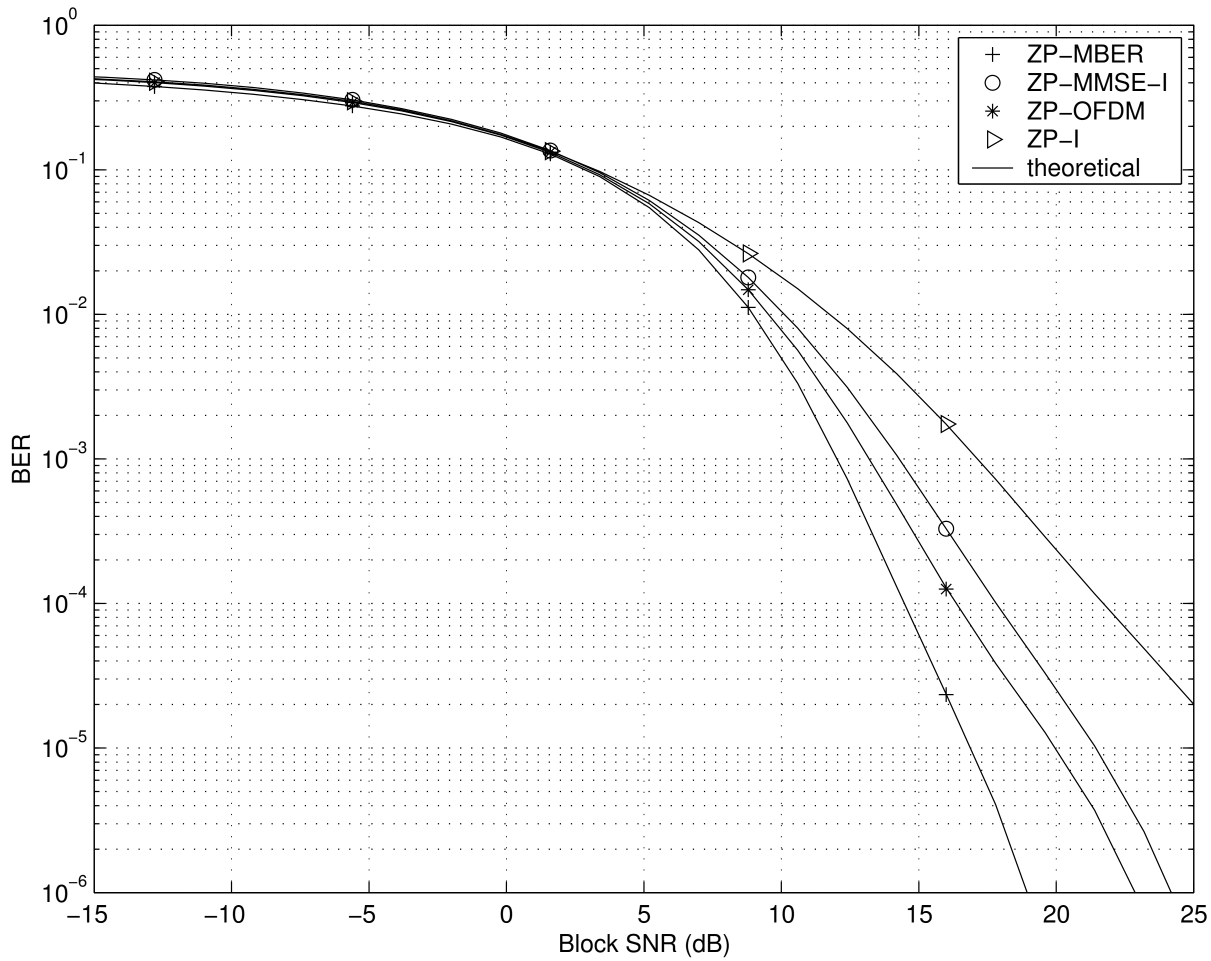
Linear MMSE equalization

- Linear MMSE equalizer: $\boldsymbol{G} = \boldsymbol{F}^H \boldsymbol{H}^H \left(\boldsymbol{R}_{vv} + \boldsymbol{H}\boldsymbol{F}\boldsymbol{F}^H \boldsymbol{H}^H \right)^{-1}$
- Now residual ISI in $\tilde{\boldsymbol{s}}$:

$$\tilde{\boldsymbol{s}} = \mathrm{Diag}(\boldsymbol{G}\boldsymbol{H}\boldsymbol{F})\boldsymbol{s} + \mathrm{OffDiag}(\boldsymbol{G}\boldsymbol{H}\boldsymbol{F})\boldsymbol{s} + \boldsymbol{G}\boldsymbol{v}$$

- However, as M gets large, ISI is almost surely Gaussian
- Algebra more difficult, but theme is the same:
  - Minimum BER precoder is an MMSE precoder with the unitary degree of freedom $\boldsymbol{V}$ chosen so that $\left|[\boldsymbol{V}]_{ij}\right| = 1/\sqrt{M}$.
  - Performance gains of the same order

BER (ave'd): MMSE equalization, ZP



ZF/MMSE Decision Feedback Equalizers

- We have considered schemes with fixed detection order
- We have found a computationally-efficient algorithm for an optimal precoder
- However, $\boldsymbol{F} \propto \boldsymbol{I}$ is often close to being optimal, hence performance gains lower

Channel unknown at transmitter

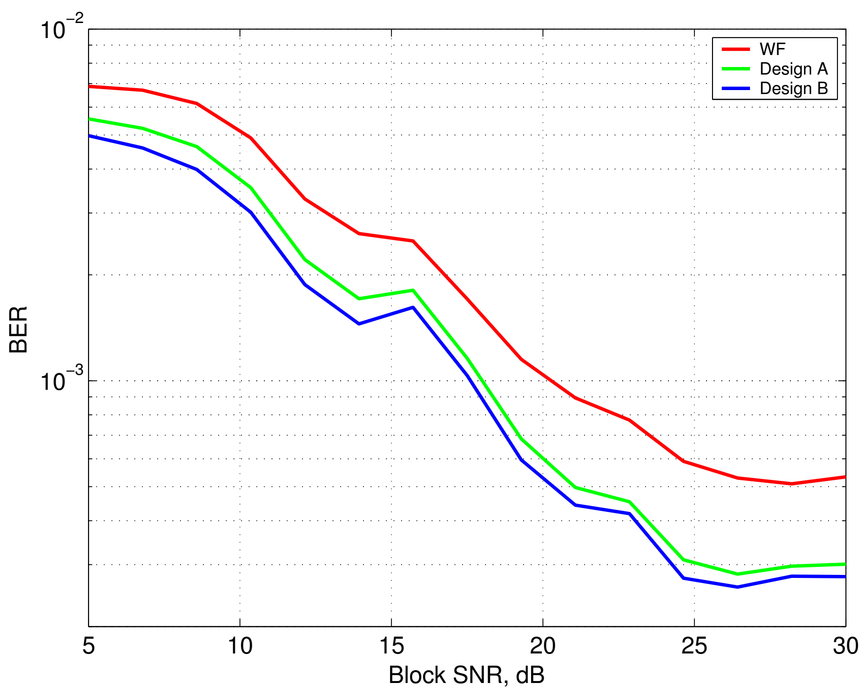
- with perfect channel knowledge at receiver
- $\boldsymbol{H}$ is unknown, but structured
- $\boldsymbol{F} \propto \boldsymbol{I}$ is optimal
  - for CP schemes with linear equalization, in sense of minimum BER (Lin and Phoong)
  - for ZP schemes with ML detection, in sense of minimizing Chernoff bound on pairwise error probability (Giannakis + students)
  - for ZP schemes with ML detection, in sense of minimizing worst-case averaged pairwise error probability (Zhang, Davidson, Wong)
- if $\boldsymbol{H}$ unknown and unstructured (space-time case) any unitary $\boldsymbol{F}$ is optimal in the PEP sense

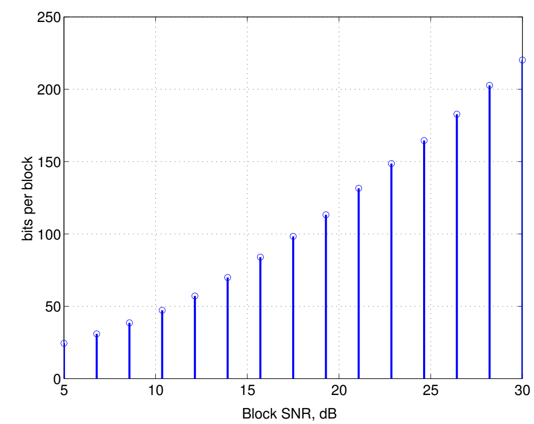
Rate Adaptive Design

- Today's focus has been on performance objectives for uniform constellations
  - Precoder tends to compensate for low-gain sub-channels
- In practice, channel knowledge at the transmitter allows joint constellation assignment and precoding (power loading) via water-filling
  - Precoder tends to allocate power and bits to high-gain sub-channels
- However, achieving reliable performance at rates promised by water-filling requires ideal Gaussian codes
- In practice, constellations are assigned by rounding to a small set
- Once that is done one ought to optimize precoder for performance
  - Several ad-hoc methods available. Reasonable performance
  - Can also apply minimum BER technique to groups of sub-channels assigned the same constellation: Significant SNR gains

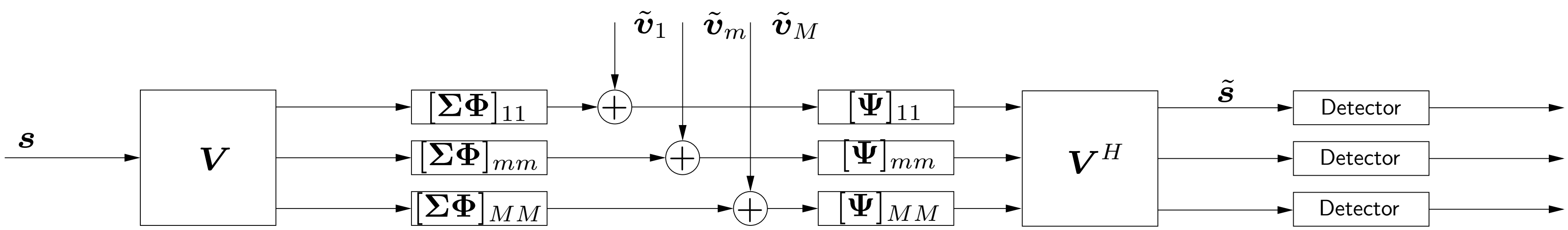
Rate Adaptive Design II

- $(M, L, P) = (32, 4, 36)$; iid Gaussian channel taps with normalization; 50 realizations;
- SNR 'Gap' 8 dB; square QAM;





Speculation: Implications for coded systems



- Our design minimizes BER of uncoded system
- What are implications for coded systems?
- Choice of $V = D$ makes all sub-channel SNRs equal.
  - Hence single code systems should perform well, at least for hard decision schemes
- Choice of $V = D$ correlates outputs
  - Hence elementwise detection and decoding is suboptimal; Complexity of optimal?
  - For belief propagation decoders, correlation between sub-channels may create short loops in graphs, and may lead to inferior performance

Conclusion

- Analytic expression for minimum BER precoder for block transmission with uniform constellation and zero-forcing equalization.
- Valid at moderate to high SNRs
- Is a special MMSE precoder
- Possible extensions abound. Some completed, some being considered
- In particular, an adaption of the idea to rate-adaptive design provides considerable SNR gain