Decomposition Example

LP example

consider LP with variables are u and v

$$\begin{array}{ll}\text{minimize} & c^T u + \tilde{c}^T v \\ \text{subject to} & Au \preceq b \\ & \tilde{A} v \preceq \tilde{b} \\ & Fu + \tilde{F} v \preceq h\end{array}$$

$Fu + \tilde{F}v \preceq h$ is the **coupling constraint**; removing it allows problem to be solved via two separate LPs

Primal decomposition

introduce variable z, and express $Fu + \tilde{F}v \preceq h$ as

$$Fu \preceq z, \qquad \tilde{F}v \preceq h - z$$

z sets the allocation of resources between the two subproblems

original problem is equivalent to **master problem**

$$\text{minimize}_z \quad \phi(z) + \tilde{\phi}(z)$$

where

$$\begin{array}{rcl} \phi(z) & = & \inf_u \ \{c^T u \mid Au \preceq b,\ Fu \preceq z\} \\ \tilde{\phi}(z) & = & \inf_v \ \{\tilde{c}^T v \mid \tilde{A}v \preceq b,\ \tilde{F}v \preceq h - z\} \end{array}$$

we can evaluate $\phi(z)$ and $\tilde{\phi}(z)$ **in parallel** by solving two LPs

to evaluate a subgradient of $\phi(z) + \tilde{\phi}(z)$:

- find λ, an optimal dual variable for first LP subproblem, for constraint $Fu \preceq z$
- find $\tilde{\lambda}$, an optimal dual variable for second LP subproblem, for constraint $\tilde{F}v \preceq h - z$
- then $g = -\lambda + \tilde{\lambda} \in \partial(\phi(z) + \tilde{\phi}(z))$

we can use subgradient method (or ellipsoid, ACCPM, . . .) to solve master problem

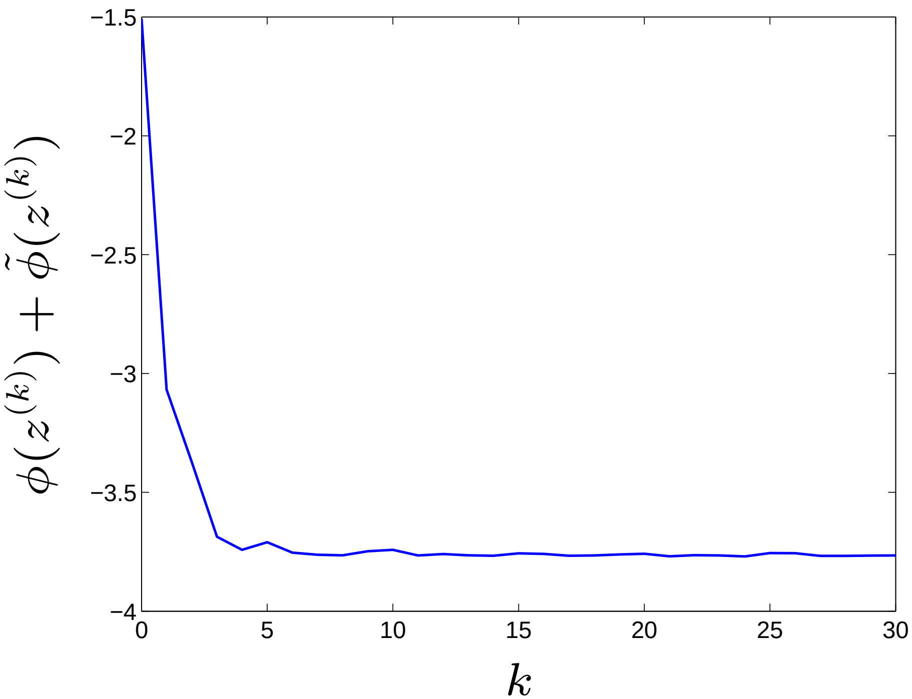
Interpretation

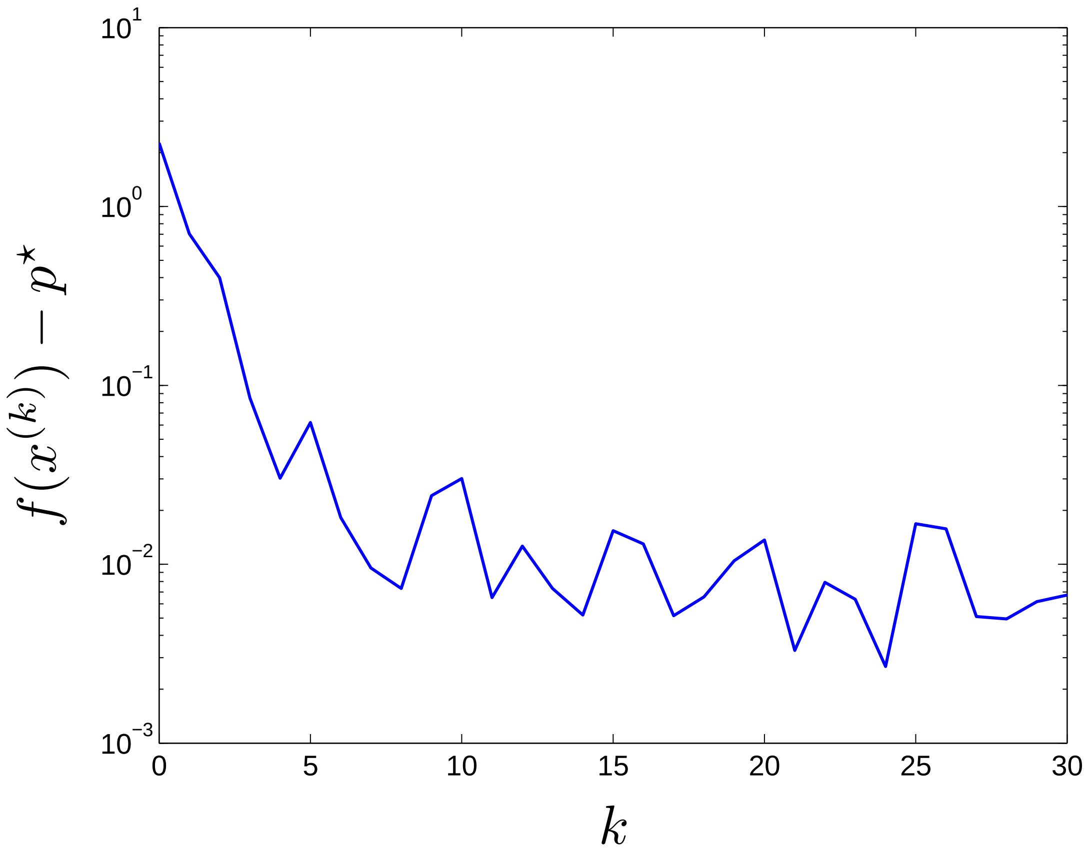
- z fixes the allocation of resources between two subproblems
- master problem iteratively finds best allocation of resources
- at each step (in subgradient method), more of each resource is allocated to the subproblem with larger Lagrange multiplier

Example

$n_u = n_v = 10$ variables, $m_u = m_v = 100$ private inequalities, $p = 5$ complicating inequalities

master problem solved by subgradient method, diminishing step size $\alpha_k = 0.1/\sqrt{k}$





10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
5
10
15
20
25
30
$f(x^{(k)}) - p^\star$
k

Dual decomposition

form partial Lagrangian, introducing Lagrange multipliers only for the coupling constraint $Fu + \tilde{F}v \preceq h$:

$$\begin{aligned} L(u,v,\lambda) &= c^T u + \tilde{c}^T v + \lambda^T (Fu + \tilde{F}v - h) \\ &= (F^T\lambda + c)^T u + (\tilde{F}^T\lambda + \tilde{c})^T v - \lambda^T h \end{aligned}$$

dual function is

$$\begin{aligned} q(\lambda) &= \inf_{u,v} \; \{L(u,v,\lambda) \mid Au \preceq b, \; \tilde{A}v \preceq \tilde{b}\} \\ &= -\lambda^T h + \inf_{Au \preceq b} (F^T\lambda + c)^T u + \inf_{\tilde{A}v \preceq \tilde{b}} (\tilde{F}^T\lambda + \tilde{c})^T v \end{aligned}$$

dual problem is

$$\begin{array}{ll}\text{maximize} & q(\lambda)\\ \text{subject to} & \lambda \succeq 0\end{array}$$

we solve this problem using the projected subgradient method

to find a subgradient of $-q$ at λ:

- let $\bar{u}$ and $\bar{v}$ be optimal solutions of the LP subproblems

$$\begin{array}{ll}\text{minimize} & (F^T\lambda + c)^T u\\ \text{subject to} & Au \preceq b\end{array} \qquad \begin{array}{ll}\text{minimize} & (\tilde{F}^T\lambda + \tilde{c})^T v\\ \text{subject to} & \tilde{A}v \preceq \tilde{b}\end{array}$$

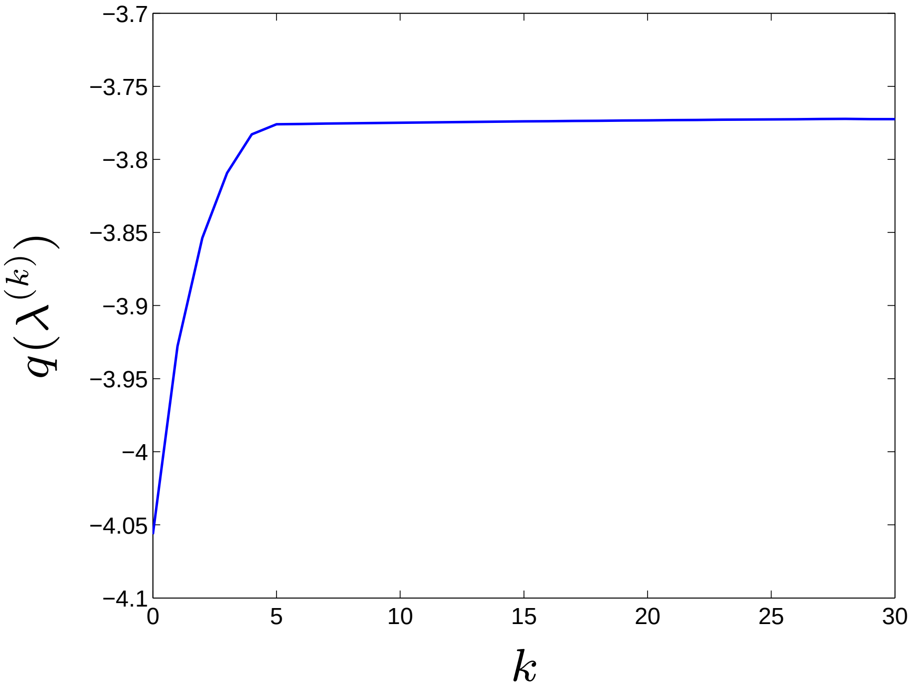
- then $g = -F\bar{u} - \tilde{F}\bar{v} + h \in \partial(-q(\lambda))$

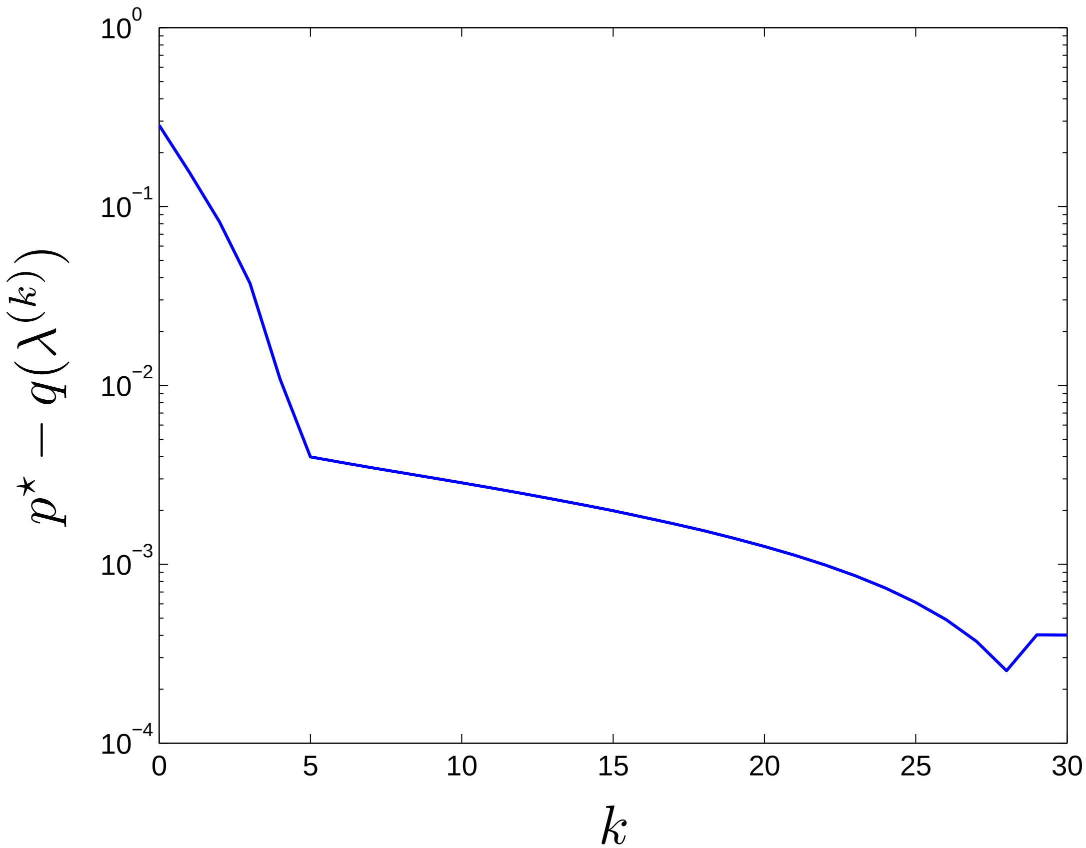
Interpretation

- λ gives *prices* of resources
- subproblems are solved separately, taking income/expense from resource usage into account
- master algorithm adjusts prices
- prices on over-subscribed resources are increased; prices on undersubscribed resources are reduced, but never negative

(Same) example

subgradient method, step size $\alpha_k = 1/\sqrt{k}$




10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
5
10
15
20
25
30
k
$p^\star - q(\lambda^{(k)})$