Alternating Projections

- alternating projection algorithm
- convergence results
- example: PSD matrix completion
- example: relaxation method for linear inequalities

Alternating projection algorithm

C, D closed convex sets in $\mathbf{R}^n$; goal is to find point in $C \cap D$

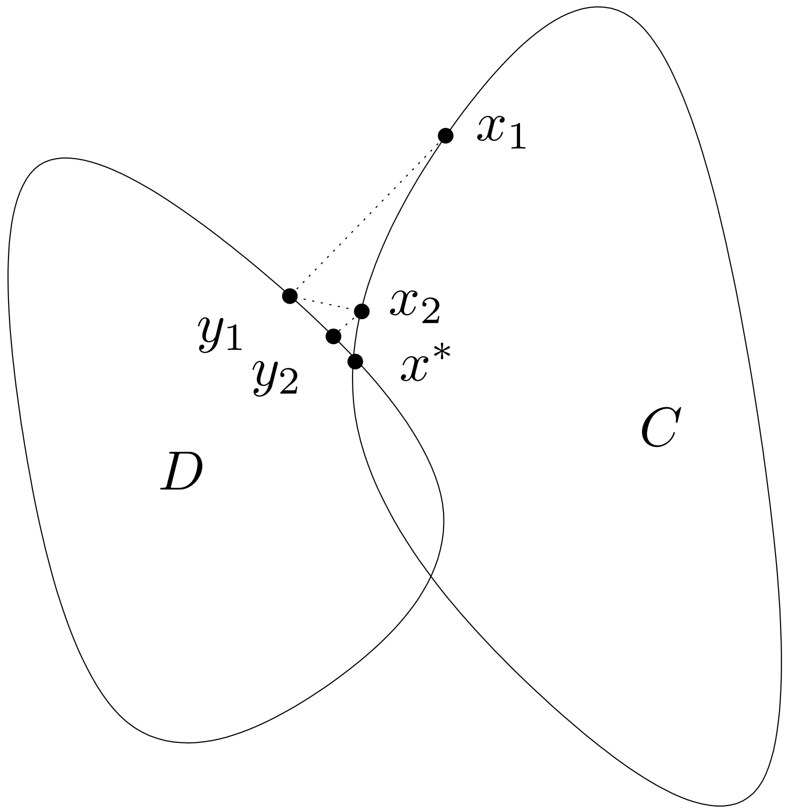
let P_C, P_D denote projection onto C and D

- start with any $x_0 \in C$
- alternately project onto C and D:

$$y_k = P_D(x_k), \qquad x_{k+1} = P_C(y_k)$$

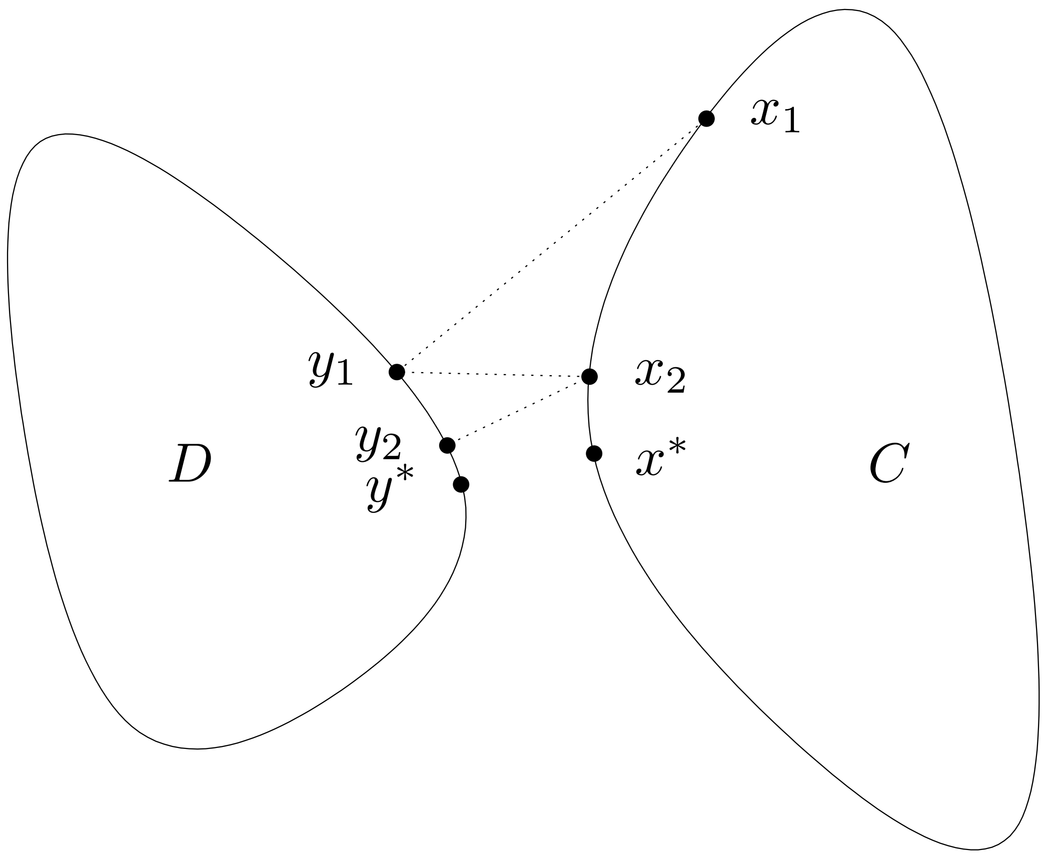
generates sequence $x_k \in C$, $y_k \in D$

first few iterations for case $C \cap D \neq \emptyset$:



. . . suggests x_k, y_k converge to a point $x^* \in C \cap D$

first few iterations for case $C \cap D = \emptyset$:



. . . suggests $x_k \to x^*$, $y_k \to y^*$, with $\|x^* - y^*\|_2 = \mathbf{dist}(C, D)$

Convergence results

- if $C \cap D \neq \emptyset$, x_k and y_k both converge to a point $x^\star \in C \cap D$ (Cheney and Goldstein, 1959)

- if $C \cap D = \emptyset$ and $\mathbf{dist}(C, D)$ is achieved, then $x_k \to x^\star \in C$, $y_k \to y^\star \in D$, where $\|x^\star - y^\star\|_2 = \mathbf{dist}(C, D)$

many generalizations, *e.g.*, sequential projection onto $k > 2$ sets, . . .

Example: Positive semidefinite matrix completion

- some entries of matrix in $\mathbf{S}^n$ fixed; find values for others so completed matrix is PSD
- $C = \mathbf{S}^n_+$, D is (affine) set in $\mathbf{S}^n$ with specified fixed entries
- projection onto C by eigenvalue decomposition, truncation: if $Y_k = \sum_{i=1}^n \lambda_i q_i q_i^T$,

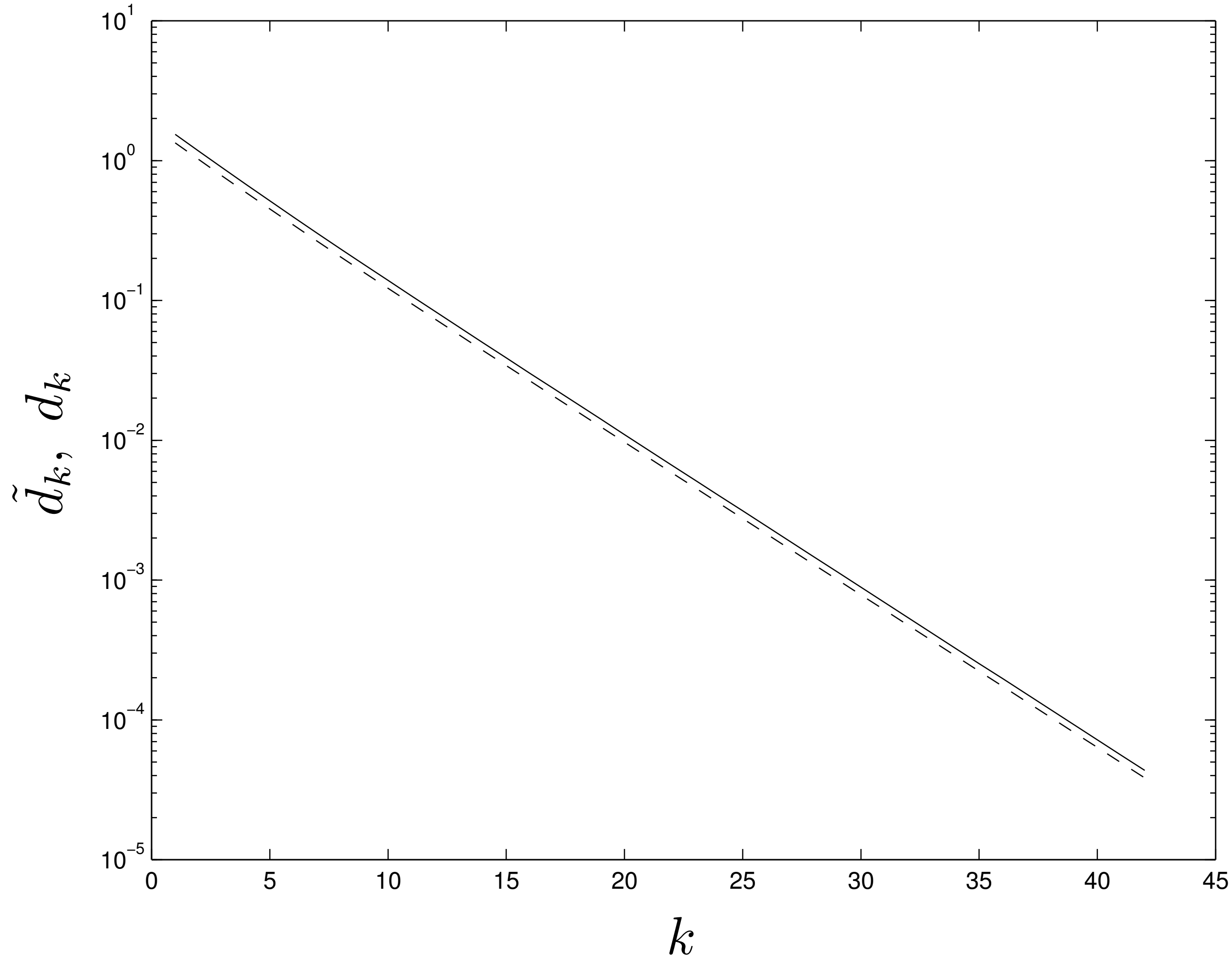
$$P_C(Y_k) = \sum_{i=1}^n \max\{0, \lambda_i\} q_i q_i^T$$

- projection of X_k onto D by re-setting specified entries to fixed values specific example:

$$X = \begin{bmatrix} 4 & 3 & ? & 2 \\ 3 & 4 & 3 & ? \\ ? & 3 & 4 & 3 \\ 2 & ? & 3 & 4 \end{bmatrix}$$

- initialize with $Y_0 = X$, with ? entries set to 0
- Y_k have correct fixed entries; X_k are PSD
- $d_k = \|X_k - Y_{k-1}\|_F$ is distance from Y_{k-1} to PSD cone
- $\tilde{d}_k = \|Y_k - X_k\|_F$ is norm of error in fixed entries

convergence is linear:



Relaxation method for linear inequalities

(Agmon 1954) find a point in (non-empty) polyhedron

$$\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$$

use sequential projection onto the m halfspaces $a_i^T x \le b_i$

- projection of z onto halfspace $a_i^T x \le b_i$ is

$$P_i(z) = \begin{cases} z & a_i^T z \le b_i \\ z - (a_i^T z - b_i) a_i & a_i^T z > b_i \end{cases}$$

- cycle through these projections to find point in $\mathcal{P}$

Example

find point in $\mathcal{P} \subseteq \mathbf{R}^{100}$, $m = 1000$ inequalities, $\|a_i\|_2 = 1$

maximum constraint violation: $r_k = \max_{i=1,\ldots,m} \ \max\{0, a_i^T x_k - b_i\}$