MR Imaging Parallel Reconstruction for Undersampled Fourier Space

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Abstract

Magnetic Resonance Imaging (MRI) requires data that is free from motion artifacts to ensure accurate diagnostics and analysis. Parallel imaging techniques such as Generalized Autocalibrating Partially Parallel Acquisitions (GRAPPA) and Sensitivity Encoding (SENSE) are widely used to accelerate image encoding by leveraging an array of receiver coils to undersample images in k-space. These techniques improve scantime efficiency, allowing for higher resolution imaging with less corruption from subject motion.

While GRAPPA is robust to coil sensitivity miscalibrations, noisy environments, and subject motion, it lacks the ability to optimize the signal-to-noise-ratio (SNR). SENSE, on the other hand, is SNR-optimal and has the ability to incorporate image priors to regularize image reconstruction, but is highly susceptible to coil sensitivity misestimation and motion artifacts. This study compares the reconstruction performance of GRAPPA and SENSE under various undersampling conditions, evaluating their robustness to noise and aliasing artifacts.

1. Introduction

Magnetic Resonance Imaging (MRI) is a non-invasive imaging technique used to obtain high-resolution images of internal structures. MR images are acquired through the use of a magnetic field to align protons in the object to be imaged, a radiofrequency pulse to move protons from alignment, and a receiving coil to pick up the signals of the protons as they realign. This process often results in extended acquisition times, increasing susceptibility to motion artifacts, noise, and patient discomfort [1].

Parallel imaging techniques improve these limitations by enabling accelerated image acquisition through undersampling in Fourier space. Generalized Autocalibrating Partially Parallel Acquisitions (GRAPPA) and Sensitivity Encoding (SENSE) are two widely used parallel imaging reconstruction methods.

GRAPPA estimates missing k-space lines by utilizing information from nearby acquired lines, making it inherently robust in noisy or low-signal environments. However, it does not optimize the signal-to-noise ratio (SNR) or incorporate image priors, which could otherwise enhance reconstruction quality. In contrast, SENSE utilizes coil sensitivity maps to directly solve for missing spatial information. SENSE optimizes SNR and allows for the incorporation of image priors. Despite these advantages, SENSE is more susceptible to noise amplification and reconstruction artifacts, particularly at high acceleration factors or when coil sensitivity estimations are inaccurate [2].

This project implements both GRAPPA and SENSE and compares their reconstruction performance of undersampled data. The evaluation focuses on image quality, assessing artifact presence, and peak signal-to-noise ratio (PSNR) comparisons to quantify reconstruction accuracy.

2. Related Work

Previous research has explored various approaches to improving MRI reconstruction, particularly by utilizing parallel imaging techniques to accelerate acquisition speeds. One key area of related work is using priors to enhance reconstruction quality in both GRAPPA and SENSE.

With SENSE, several methods have been explored to incorporate prior information to improve reconstruction accuracy. Regularization techniques, such as total variation (TV) minimization, Tikhonov, and Bremen integration, have been applied to suppress noise amplification and improve SNR. Additionally, advanced reconstruction methods employ machine learning-based priors, particularly deep learning models trained on high-quality MRI data. These improvements better estimate missing frequencies and reduce aliasing artifacts compared to the GRAPPA reconstruction algorithm implemented in this project [3].

In GRAPPA, notable work involves incorporating prior information to improve robustness and reconstruction fidelity. Studies have proposed integrating Tikhonov regularization into the GRAPPA reconstruction process, leveraging information from the Autocalibrating Signal (ACS) region to enhance stability and image quality. This approach helps to mitigate noise and reduce artifacts that commonly arise in accelerated acquisitions. Another significant development in parallel imaging reconstruction is the Bayesian framework that combines GRAPPA and SENSE by utilizing prior distributions derived from k-space data. This probabilistic approach improves missing spatial frequency estimation, effectively reducing aliasing artifacts and enabling full-field-of-view image reconstruction [4][5].

Recent work has also demonstrated that generative priors can further refine MRI reconstructions. By enforcing learned constraints on the reconstructed images, these priors help mitigate artifacts and improve the overall fidelity of undersampled k-space data reconstructions. The use of diffusion models in image reconstruction also improves the robustness of parallel imaging methods [6].

Beyond GRAPPA and SENSE, another parallel imaging technique is Iterative Self-Consistent Parallel Imaging Reconstruction (SPIRiT). Unlike GRAPPA, which explicitly estimates missing k-space lines, and SENSE, which directly solves for the image domain, SPIRiT enforces consistency across acquired and estimated k-space data through an iterative process. This method has been found to provide improved image quality and robustness, particularly in high-acceleration scenarios [7].

3. Method and Theory

3.1. Accelerating k-space acquisition

MRI relies on the acquisition of data in k-space, which represents the frequency content of the subject's spatial information. By nature of the MRI signal, k-space must be sampled in a sequential manner using precisely controlled magnetic field gradients and radiofrequency pulses. This process results in relatively long scan times on the order of tens of minutes, which compromises patient comfort, increases susceptibility to motion artifacts, and limits the feasibility of imaging techniques that capture dynamic real-time information. For these reasons, developing image reconstruction techniques that recover accurate images from accelerated k-space acquisitions has become a central research focus.

In this project, we implement reconstruction methods (Figure 1) designed to produce high-quality images from k-space acquisitions that have been accelerated via undersampling and received by an array of receiver coils. These coils have distinct spatial sensitivity profiles that enable the reconstruction algorithms to discern spatial information from points that overlap in the aliased, undersampled image.

3.2. SENSE reconstruction

Sensitivity encoding reconstruction (SENSE) is a parallel imaging technique that exploits receiver coil spatial sensitivity profiles to reconstruct MRI images from undersampled data.

Let R represent the undersampling factor of the accelerated imaging sequence (i.e. one out of every R samples of fully-sampled k-space is collected) and let N represent the number of receiver coils. When this data is inverse Fourier transformed into the image domain, R pixels from the true image alias into a single pixel. We model the measured pixel value as a linear combination of the R aliased pixels weighted by the coil sensitivities at each of the aliased pixels. For coils indexed by $i = 1, ..., N$, the signal recorded by the ith coil is

$$y_i = \sum_{j=1}^{R} C_i(\mathbf{x}_j) m(\mathbf{x}_j) + \eta_i$$

where $C_i(\mathbf{x}_j)$ is the sensitivity of coil i at pixel $\mathbf{x}_j$, $m(\mathbf{x}_j)$ is the true image value at pixel $\mathbf{x}_j$, and η_i is the noise in the ith coil's measurement. In matrix-vector notation, we write this compactly as

$$\mathbf{y} = \mathbf{C}\mathbf{m} + \boldsymbol{\eta}$$

where

- $\mathbf{y} \in \mathbb{C}^N$ are the aliased coil readings
- $\mathbf{C} \in \mathbb{C}^{N \times R}$ is the coil sensitivity matrix
- $\mathbf{m} \in \mathbb{C}^R$ are the true pixel values contributing to the aliased point
- $\boldsymbol{\eta} \in \mathbb{C}^N$ is the measurement noise

We formulate the reconstruction as minimizing the squared error between the measurements and the forward model

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} ||\mathbf{C}\mathbf{m} - \mathbf{y}||_2^2 = (\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\mathbf{y}$$

Because SENSE reconstruction is formulated as a maximum likelihood estimation of the underlying image, it is the SNR-optimal reconstruction method for the accelerated image given that the model assumptions (i.e. Gaussian uncorrelated noise and correct coil sensitivity maps) are correct. Coil sensitivity maps must be estimated for each patient because they depend on factors such as patient position and tissue properties. Mis-estimated coil sensitivities introduce an error term in the $\mathbf{C}$ matrix, which is amplified by the inversion in the least squares estimation. Thus, while SENSE is SNR-optimal, it is also sensitive to small errors in the model formulation.

3.3. GRAPPA reconstruction

Generalized Autocalibrating Partial Parallel Acquisition (GRAPPA) is a reconstruction technique that enables accelerated MRI by synthesizing missing k-space data. Unlike the SENSE technique, GRAPPA avoids the need for

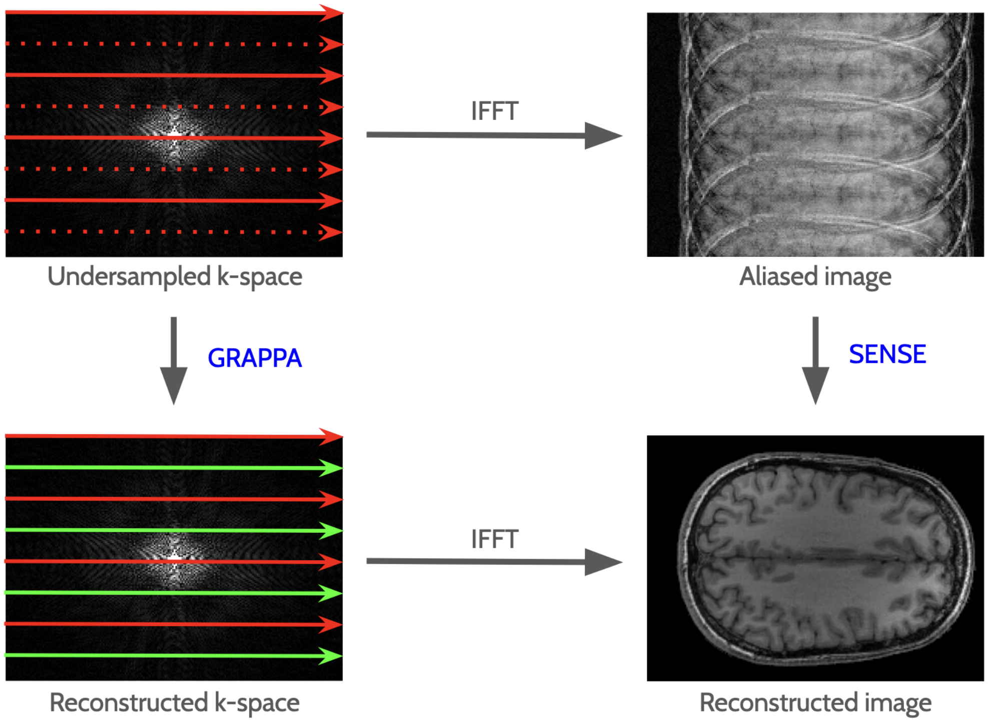


Figure 1: The reconstruction pipeline implemented in this project follows two branches from the input undersampled k-space to the output reconstructed image. GRAPPA operates in the Fourier domain, synthesizing the missing k-space lines (green) from the undersampled MRI sequence. SENSE operates in the image domain, using coil sensitivity maps to unalias the undersampled image.

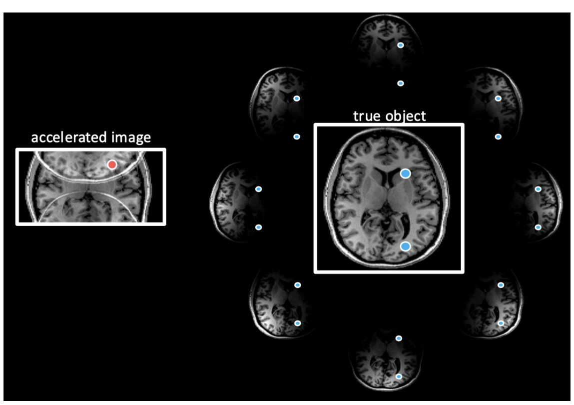


Figure 2: An $R = 2$ case is shown here with eight coils. The red point in the accelerated image contains contributions from two points (blue) in the true image. Each pair of aliased points is weighted differently in each of the receiver coil channels (shown surrounding the object). Though only the combined accelerated image is shown here, there is an accelerated image for each coil, thus the problem is a linear system with eight equations and two unknowns. [3]

estimated spatially varying coil sensitivity maps. Instead, GRAPPA requires a small contiguous autocalibration region of fully sampled k-space for calibration (Figure 3).

Let $M^{(i)}_{miss}(\mathbf{k})$ be the k-space value to be estimated of a missing sample from coil i (e.g. due to undersampling). The GRAPPA reconstruction assumes that $M^{(i)}_{miss}(\mathbf{k})$ can be estimated as a linear combination of nearby acquired samples:

$$M^{(i)}_{miss}(\mathbf{k}) = \sum_{c=1}^{N} \sum_{(m,n)\in W} w_{(c,m,n)} M_c(k_x + m, k_y + n)$$

where

- $c = 1, ..., N$ indexes the receiver coils
- W represents a neighborhood of acquired points around the missing point
- $w_{(c,m,n)}$ are weights estimated from the autocalibration data
- $M_c(\mathbf{k})$ represents the acquired k-space value from coil c at $\mathbf{k} = (k_x, k_y)$

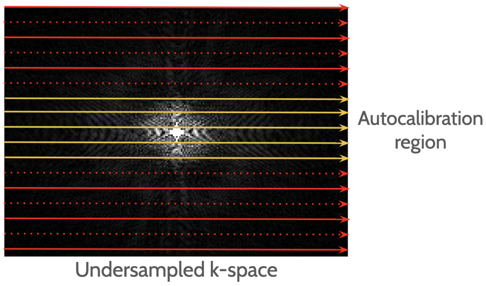


Figure 3: The autocalibration region is a small continuous region of fully sampled k-space in an otherwise undersampled acquisition

In order to determine the weights $w_{(c,m,n)}$, the GRAPPA algorithm solves a linear system that fits the autocalibration signal region (ACS) data by relating a neighborhood of k-space samples to a target point. The system for coil i is written as

$$\mathbf{M}^{(i)} = \mathbf{A}\mathbf{w}^{(i)}$$

where

- $\mathbf{M}^{(i)} \in \mathbb{C}^K$ are the target k-space points in coil i corresponding to the K points in the ACS
- $\mathbf{A} \in \mathbb{C}^{K\times L}$ are the ACS k-space samples that contribute to the target point, where $L = N|W|$ is the total number of samples that contribute to the weighted combination.
- $\mathbf{w}^{(\mathbf{i})} \in \mathbb{C}^L$ represents the weights

The GRAPPA algorithm then solves for the estimated weights by solving the least squares problem

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}} ||\mathbf{A}\mathbf{w}^{(i)} - \mathbf{M}^{(i)}||_2^2 = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{M}^{(i)}$$

These weights then form a kernel that is convolved with the accelerated k-space acquisition to synthesize the missing data(Figure 4).

4. Analysis, Evaluation, and Comparison to Other Methods

When comparing GRAPPA and SENSE, it is important to look at the differences in the reconstruction algorithms. GRAPPA relies on interpolation weights that are learned from the fully sampled k-space data. In accelerated imaging, the weights are applied to the undersampled data to estimate the missing k-space lines. GRAPPA is compatible with, but does not require coil sensitivity maps for the reconstruction, making it more robust than SENSE, especially when working with incomplete or incorrect data.

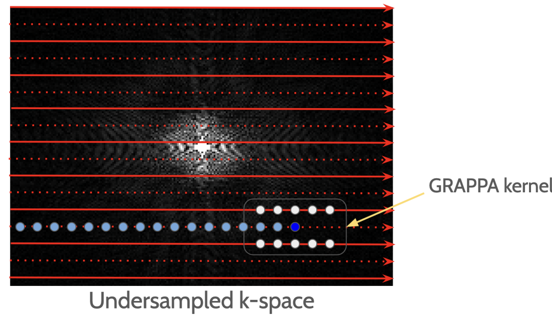


Figure 4: The undersampled k-space data is convolved with the estimated weighting kernel to synthesize missing data.

GRAPPA is not inherently SNR-optimal. GRAPPA handles noise and motion artifacts well due to its data-driven interpolation approach, but it does not leverage the coil sensitivity profiles to optimize SNR. This results in typically lower SNR values than SENSE, particularly when working in noisy environments or with higher acceleration factors.

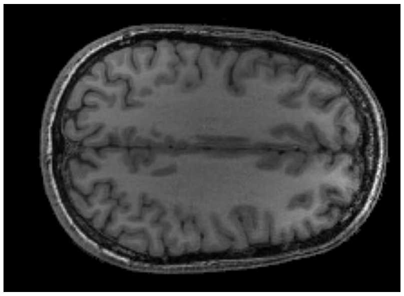
SENSE operates by combining undersampled data from all coils using their respective sensitivity maps to solve for the true image. With accurate coil sensitivity maps, SENSE effectively reduces noise and improve image quality by optimally weighting the data from each coil. Any misestimation or calibration errors in the sensitivity maps can introduce significant reconstruction artifacts since this method depends on accurate sensitivity maps. At higher acceleration factors, SENSE's SNR performance degrades because any inaccuracies in the sensitivity maps become more pronounced which amplifys reconstruction errors.

5. Results

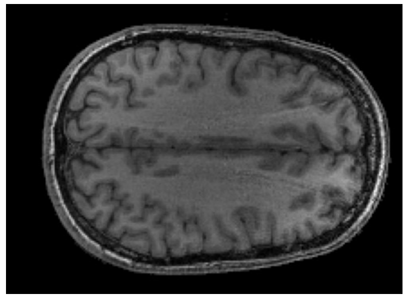
A comparison of image reconstructions using both SENSE and GRAPPA under various acceleration factors is presented in Figure 5. The Figure illustrates the reconstruction quality at acceleration factors of 2 and 4 along the vertical direction, which signifies undersampling by a factor of 2 and 4. The reconstructed images are compared through a qualitative inspection of the presence of artifacts or aliasing, along with a quantitative comparison of the PSNR values for each method.

The ground truth image was reconstructed with the sum of squares method from fully-sampled k-space data. This method combines data from the coils by taking the square root of the sum of the squares of the individual coil images. The ground truth image obtained from this reconstruction was used in the PSNR calculation for the undersampled reconstructed images.

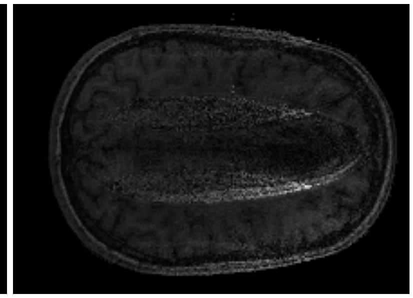
At $R_x = 2$, both SENSE and GRAPPA provide reasonable reconstructions. SENSE produces a PSNR of 35.81



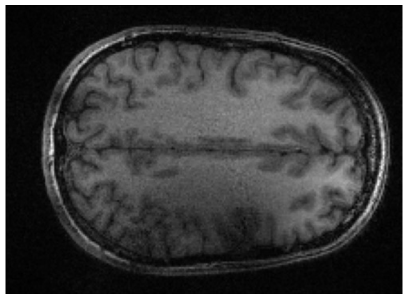
(a) Ground Truth



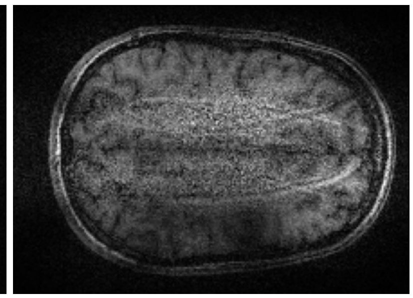
(b) SENSE, $R_x = 2$
PSNR = 35.81 dB



(c) SENSE, $R_x = 4$
PSNR = 15.53 dB



(d) GRAPPA, $R_x = 2$
PSNR = 33.33 dB



(e) GRAPPA, $R_x = 4$
PSNR = 24.10 dB

Figure 5: Comparison of reconstructed images. We used a convolution kernel with dimensions 5x2 in both GRAPPA reconstructions.

dB, and provides good image quality with minimal artifacts. In contrast, GRAPPA reconstructs the image with a PSNR of 33.33 dB, which is slightly lower than SENSE but provides a higher quality image due to the lack of visible aliasing.

At $R_x = 4$, the difference in performance between SENSE and GRAPPA is evident. SENSE drops in PSNR to 15.53 dB, and produces a significantly degraded image. The reconstructed image has poor contrast and is very dark. The drop in PSNR is largely due to its vulnerability to noise amplification at higher acceleration factors, along with the sensitivity of SENSE to estimations in coil sensitivities. At $R_x = 4$, GRAPPA produced a PSNR of 24.10 dB, which indicates a more robust reconstruction than SENSE at higher accelerations. The image has visible aliasing artifacts but is sharper and brighter than the SENSE $R_x = 4$ reconstructed image. Although the image quality at $R_x = 4$ is reduced for both methods compared to $R_x = 2$, GRAPPA demonstrates better resilience to noise and artifact propagation.

These results highlight the trade-offs between the two methods. SENSE provides high-quality reconstructions when coil sensitivity maps are accurate, but its performance degrades at higher acceleration factors. GRAPPA does not optimize SNR as effectively as SENSE, but it produces more robust reconstructions across various acceleration factors and is less susceptible to noise.

6. Discussion

This project explored the strengths and limitations of both GRAPPA and SENSE for parallel image reconstruction. While GRAPPA provides robust performance in the presence of coil miscalibration and subject motion, it lacks the ability to utilize additional information through image priors. SENSE provides better SNR optimization and provides a convenient template for incorporating image priors:

$$\hat{\mathbf{m}} = \arg\min_{\mathbf{m}} ||\mathbf{Cm} - \mathbf{y}||_2^2 + \lambda \mathcal{R}(\mathbf{m})$$

where the $\mathcal{R}(\mathbf{m})$ can be designed to penalize deviations from expected attributes of the underlying image $\mathbf{m}$. However, SENSE had the downside of image degradation in the presence of coil misestimations or at high acceleration factors.

One next step for this project is to incorporate autocalibration data as a prior in the GRAPPA reconstruction process.

Autocalibration data, which is obtained from a small portion of the fully-sampled k-space, represents the Fourier coefficients of a low resolution version of the underlying image. Incorporating this low resolution image as a prior in teh reconstruction estimate could improve the quality of the GRAPPA-reconstructed image [9]. Once the weights $\mathbf{w}^{(i)} = \mathbf{A}^{\dagger}\mathbf{M}^{(i)}$ are calculated, the image reconstruction step can be represented as a minimization problem:

$$\hat{\mathbf{M}}_{\text{all}}^{(i)} = \underset{\mathbf{M}_{\text{all}}^{(i)}}{\arg\min} ||\mathbf{A}^{\dagger}\mathbf{M}_{\text{all}}^{(i)} - \mathbf{w}^{(i)}||_2^2 + \lambda||\mathcal{F}^{-1}(\mathbf{M}_{\text{all}}^{(i)}) - \mathbf{m}_{\text{acs}}||_2^2$$

where the first term enforces that the reconstructed k-space is consistent with the calculated weights and the second term enforces that the reconstructed k-space data $\mathbf{M}_{\text{all}}^{(i)}$ is consistent with the low-resolution image $\mathbf{m}_{\text{acs}}$ produced by the ACS data.

The use of autocalibration data as a prior can help bridge the gap between the robustness of GRAPPA and the flexibility of SENSE.

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