EE365: Linear Quadratic Trading Example
Linear quadratic trading: Dynamics

- $x_{t+1} = f_t(x_t, u_t, \rho_t) = \text{diag}(\rho_t)(x_t + u_t)$
- $x_t \in \mathbb{R}^n$ is dollar amount of holding in $n$ assets
- $(x_t)_i < 0$ means short position in asset $i$ in period $t$
- $u_t \in \mathbb{R}^n$ is dollar amount of each asset bought at beginning of period $t$
- $(u_t)_i < 0$ means asset $i$ is sold in period $t$
- $x_t^+ = x_t + u_t$ is post-trade portfolio
- $\rho_t \in \mathbb{R}_{++}^n$ is (random) return of assets over period $(t, t + 1]$
- returns independent, with $\mathbb{E} \rho_t = \bar{\rho}_t$, $\mathbb{E} \rho_t \rho_t^T = \Sigma_t$
Linear quadratic trading: Stage cost

stage cost for $t = 0, \ldots, T - 1$ is (convex quadratic)

$$g_t(x, u) = 1^T u + \frac{1}{2} (\kappa_t u^2 + \gamma (x + u)^T Q_t (x + u))$$

with $Q_t > 0$

- first term is gross cash in
- second term is quadratic transaction cost (square is elementwise; $\kappa_t > 0$)
- third term is risk (variance of post-trade portfolio for $Q_t = \Sigma_t - \bar{\rho}_t \bar{\rho}^T_t$)
- $\gamma > 0$ is risk aversion parameter
- minimizing total stage cost equivalent to maximizing (risk-penalized) net cash taken from portfolio
Linear quadratic trading: Terminal cost

- terminal cost: \( g_T(x) = -1^T x + \frac{1}{2} \kappa_T^T x^2, \kappa_T > 0 \)

- this is net cash in if we close out (liquidate) final positions, with quadratic transaction cost
Linear quadratic trading: DP

- value functions quadratic (including linear and constant terms):
  \[ v_t(x) = \frac{1}{2} (x^T P_t x + 2q_t^T x + r_t) \]

- we’ll need formula
  \[ \mathbb{E}(\text{diag}(\rho_t)P \text{ diag}(\rho_t)) = P \circ \Sigma_t \]
  where \( \circ \) is Hadamard (element-wise) product

- optimal expected tail cost
  \[ \mathbb{E} v_{t+1}(f_t(x,u,\rho_t)) = \mathbb{E} v_{t+1}(\text{diag}(\rho_t)x^+) \]
  \[ = \frac{1}{2} ((x^+)^T P_{t+1} \circ \Sigma_t x^+ + 2q_{t+1}^T \text{ diag}(\bar{\rho}_t)x^+ + r_{t+1}) \]
Linear quadratic trading: DP

- \( P_T = \text{diag}(\kappa_T), q_T = -1, r_T = 0 \)

- recall \( v_t(x) = \min_u \mathbb{E} (g_t(x, u) + v_{t+1}(\text{diag}(\rho_t)(x + u))) \)

- for \( t = T - 1, \ldots, 0 \) we minimize over \( u \) to get optimal policy:

\[
\mu_t(x) = \arg\min_u (u^T(S_{t+1} + \text{diag}(\kappa_t))u + 2(S_{t+1}x + s_{t+1} + 1)^T u)
\]

\[
= -(S_{t+1} + \text{diag}(\kappa_t))^{-1}(S_{t+1}x + s_{t+1} + 1)
\]

\[
= K_t x + l_t
\]

where

\[
S_{t+1} = P_{t+1} \circ \Sigma_t + \gamma Q_t, \quad s_{t+1} = \bar{\rho}_t \circ q_{t+1}
\]

- using \( u = K_t x + l_t \) we then have

\[
v_t(x) = \frac{1}{2} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} S_{t+1}(I + K_t) & s_{t+1} + S_{t+1}l_t \\ s_{t+1}^T + l_t^T S_{t+1} & r_{t+1} + (s_{t+1} + 1)^T l_t \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}
\]
Linear quadratic trading: value iteration

- set $P_T = \text{diag}(\kappa_T)$, $q_T = -1$, $r_T = 0$
- for $t = T - 1, \ldots, 0$
  
  $K_t = -(S_{t+1} + \text{diag}(\kappa_t))^{-1}S_{t+1}$
  
  $l_t = -(S_{t+1} + \text{diag}(\kappa_t))^{-1}(s_{t+1} + 1)$
  
  $P_t = S_{t+1}(I + K_t)$
  
  $q_t = s_{t+1} + S_{t+1}l_t$
  
  $r_t = r_{t+1} + (s_{t+1} + 1)^Tl_t$

  where

  $S_{t+1} = P_{t+1} \circ \Sigma_t + \gamma Q_t, \quad s_{t+1} = \bar{\rho}_t \circ q_{t+1}$

- optimal policy: $\mu_t^*(x) = K_t x + l_t$
- can write as $\mu_t^*(x) = K_t(x - x_t^{\text{tar}}), \quad x_t^{\text{tar}} = -K_t^{-1}l_t = -S_{t+1}^{-1}(s_{t+1} + 1)$
- $J^* = \mathbb{E} v_0(x_0)$
Linear quadratic trading: Numerical instance

- $n = 30$ assets over $T = 100$ time-steps
- initial portfolio $x_0 = 0$
- $\bar{\rho}_t = \bar{\rho}, \Sigma_t = \Sigma$ for $t = 0, \ldots, T - 1$
- $Q_t = \Sigma - \bar{\rho}\bar{\rho}^T$ for $t = 0, \ldots, T - 1$
- asset returns log-normal, expected returns range over $\pm 3\%$ per period
- asset return standard deviations range from $0.4\%$ to $9.8\%$
- asset correlations range from $-0.3$ to $0.8$
Linear quadratic trading: Numerical instance

- $N = 100$ Monte Carlo simulations
- $J^* = v_0(x_0) = -237.5$ (Monte Carlo estimate: $-238.4$)
- exact (red), MC estimate (blue), and samples (gray); $J^*$ red dashed
Linear quadratic trading: Numerical instance

we define $x_{T+1} = 0$, i.e., we close out the position during period $T$