EE365: Linear Quadratic Stochastic Control

Continuous state Markov decision process

Affine and quadratic functions

Linear quadratic Markov decision process
Continuous state Markov decision process
Continuous state Markov decision problem

- **dynamics:** \( x_{t+1} = f_t(x_t, u_t, w_t) \)
- \( x_0, w_0, w_1, \ldots \) independent
- **stage cost:** \( g_t(x_t, u_t, w_t) \)
- **state feedback policy:** \( u_t = \mu_t(x_t) \)
- choose policy to minimize

\[
J = \mathbb{E} \left( \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T) \right)
\]

- we consider the case \( X = \mathbb{R}^n \), \( U = \mathbb{R}^m \)
Continuous state Markov decision problem

- many (mostly mathematical) pathologies can occur in this case
  - but not in the special case we’ll consider
- a basic issue: how do you even represent the functions $f_t$, $g_t$, and $\mu_t$?
  - for $n$ and $m$ very small (say, 2 or 3) we can use gridding
  - we can give the coefficients in some (dense) basis of functions
  - most generally, we assume we have a method to compute function values, given the arguments
  - exponential growth that occurs in gridding called curse of dimensionality
Continuous state Markov decision problem: Dynamic programming

- set \( v_T(x) = g_T(x) \)

- for \( t = T - 1, \ldots, 0 \)
  \[
  \mu_t(x) \in \arg\min_u E \left( g_t(x,u,w_t) + v_{t+1}(f_t(x,u,w_t)) \right)
  \]
  \[
  v_t(x) = E \left( g_t(x,\mu_t(x),w_t) + v_{t+1}(f_t(x,\mu_t(x),w_t)) \right)
  \]

- this gives value functions and optimal policy, \textit{in principle only}

- but you can’t in general represent, much less compute, \( v_t \) or \( \mu_t \)
Continuous state Markov decision problem: Dynamic programming

for DP to be tractable, \( f_t \) and \( g_t \) need to have special form for which we can

- represent \( v_t, \mu_t \) in some tractable way
- carry out expectation and minimization in DP recursion

one of the few situations where this holds: \textit{linear quadratic problems}

- \( f_t \) is an affine function of \( x_t, u_t \) (‘linear dynamical system’)
- \( g_t \) are convex quadratic functions of \( x_t, u_t \)
Linear quadratic problems

for linear quadratic problems

- value functions $v_t^*$ are quadratic
- hence representable by their coefficients
- we can carry out the expectation and the minimization in DP recursion explicitly using linear algebra
- optimal policy functions are affine: $\mu_t^*(x) = K_t x + l_t$
- we can compute the coefficients $K_t$ and $l_t$ explicitly

in other words:

we can solve linear quadratic stochastic control problems in practice
Affine and quadratic functions
Affine functions

- $f : \mathbb{R}^p \to \mathbb{R}^q$ is affine if it has the form
  \[ f(x) = Ax + b \]
  i.e., it is a linear function plus a constant

- a linear function is special case, with $b = 0$

- affine functions closed under sum, scalar multiplication, composition (with explicit formulas for coefficients in each case)
Quadratic function

- \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is quadratic if it has the form
  \[
  f(x) = \frac{1}{2} x^T P x + q^T x + \frac{1}{2} r
  \]
  with \( P = P^\top \in \mathbb{R}^{n \times n} \) (the \( \frac{1}{2} \) on \( r \) is for convenience)

- often write as quadratic form in \((x, 1)\):
  \[
  f(x) = \frac{1}{2} \begin{bmatrix} x \end{bmatrix}^\top \begin{bmatrix} P & q \\ q^\top & r \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}
  \]

- special cases:
  - quadratic form: \( q = 0, r = 0 \)
  - affine (linear) function: \( P = 0 \) (\( P = 0, r = 0 \))
  - constant: \( P = 0, q = 0 \)

- uniqueness: \( f(x) = \tilde{f}(x) \iff P = \tilde{P}, q = \tilde{q}, r = \tilde{r} \)
Calculus of quadratic functions

- quadratic functions on $\mathbb{R}^n$ form a vector space of dimension

$$\frac{n(n + 1)}{2} + n + 1$$

- i.e., they are closed under addition, scalar multiplication
Composition of quadratic and affine functions

» suppose

» $f(z) = \frac{1}{2}z^T P z + q^T z + \frac{1}{2} r$ is quadratic function on $\mathbb{R}^m$

» $g(x) = Ax + b$ is affine function from $\mathbb{R}^n$ into $\mathbb{R}^m$

» then composition $h(x) = (f \circ g)(x) = f(Ax + b)$ is quadratic

» write $h(x)$ as

\[
\frac{1}{2} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left( \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \begin{bmatrix} A & b \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix}
\]

» so matrix multiplication gives us the coefficient matrix of $h$
Convexity and nonnegativity of a quadratic function

- $f$ is convex (graph does not curve down) if and only if $P \geq 0$ (matrix inequality)

- $f$ is strictly convex (graph curves up) if and only if $P > 0$ (matrix inequality)

- $f$ is nonnegative (i.e., $f(x) \geq 0$ for all $x$) if and only if

\[
\begin{bmatrix}
P & q \\
q^T & r \\
\end{bmatrix} \geq 0
\]

- $f(x) > 0$ if and only if matrix inequality is strict

- nonnegative $\Rightarrow$ convex
Checking convexity and nonnegativity

we can check convexity or nonnegativity in $O(n^3)$ operations by eigenvalue decomposition, Cholesky factorization, . . .

composition with affine function preserves convexity, nonnegativity:

\[ f \text{ convex, } g \text{ affine } \implies f \circ g \text{ convex} \]

linear combination of convex quadratics, with nonnegative coefficients, is convex quadratic

if $f(x, w)$ is convex quadratic in $x$ for each $w$ (a random variable) then

\[ g(x) = \mathbb{E}_w f(x, w) \]

is convex quadratic (i.e., convex quadratics closed under expectation)
Minimizing a quadratic

- if $f$ is not convex, then $\min_x f(x) = -\infty$
- otherwise, $x$ minimizes $f$ if and only if $\nabla f(x) = Px + q = 0$
- for $q \not\in \text{range}(P)$, $\min_x f(x) = -\infty$
- for $P > 0$, unique minimizer is $x = -P^{-1}q$
- minimum value is
  \[
  \min_x f(x) = -\frac{1}{2}q^\top P^{-1}q + \frac{1}{2}r
  \]
  (a concave quadratic function of $q$)
- for case $P \geq 0$, $q \in \text{range}(P)$, replace $P^{-1}$ with $P^\dagger$
Partial minimization of a quadratic

- suppose $f$ is a quadratic function of $(x, u)$, convex in $u$
- then the partial minimization function

$$g(x) = \min_{u} f(x, u)$$

is a quadratic function of $x$; if $f$ is convex, so is $g$
- the minimizer $\arg\min_{u} f(x, u)$ is an affine function of $x$
- minimizing a convex quadratic function over some variables yields a convex quadratic function of the remaining ones
- i.e., convex quadratics closed under partial minimization
Partial minimization of a quadratic

\[ f(x, u) = \frac{1}{2} \begin{bmatrix} x \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} P_{xx} & P_{xu} & q_x \\ P_{ux} & P_{uu} & q_u \\ q_x^T & q_u^T & r \end{bmatrix} \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \]

with \( P_{uu} > 0, \ P_{ux} = P_{xu}^T \)

minimizer of \( f \) over \( u \) satisfies

\[ 0 = \nabla_u f(x, u) = P_{uu}u + P_{ux}x + q_u \]

so \( u = -P_{uu}^{-1}(P_{ux}x + q_u) \) is an affine function of \( x \)
Partial minimization of a quadratic

- substituting $u$ into expression for $f$ gives

$$g(x) = \frac{1}{2} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_{xx} - P_{xu} P_{uu}^{-1} P_{ux} & q_x - P_{xu} P_{uu}^{-1} q_u \\ q_x^T P_{uu}^{-1} P_{ux} & r - q_u^T P_{uu}^{-1} q_u \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

- $P_{xx} - P_{xu} P_{uu}^{-1} P_{ux}$ is the Schur complement of $P$ w.r.t. $u$

- $P_{xx} - P_{xu} P_{uu}^{-1} P_{ux} \geq 0$ if $P \geq 0$

- or simpler: $g$ is composition of $f$ with affine function $x \mapsto (x, u)$

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I \\ -P_{uu}^{-1} P_{ux} \end{bmatrix} x + \begin{bmatrix} 0 \\ -P_{uu}^{-1} q_u \end{bmatrix}$$

- we already know how to form composition quadratic (affine), and the result is convex
Summary

convex quadratics are closed under

- addition
- expectation
- pre-composition with an affine function
- partial minimization

in each case, we can explicitly compute the coefficients of the result using linear algebra
Linear quadratic Markov decision process
(Random) linear dynamical system

- dynamics \( x_{t+1} = f_t(x_t, u_t, w_t) = A_t(w_t)x_t + B_t(w_t)u_t + c_t(w_t) \)
- for each \( w_t \), \( f_t \) is affine in \( (x_t, u_t) \)
- \( x_0, w_0, w_1, \ldots \) are independent
- \( A_t(w_t) \in \mathbb{R}^{n \times n} \) is dynamics matrix
- \( B_t(w_t) \in \mathbb{R}^{n \times m} \) is input matrix
- \( c_t(w_t) \in \mathbb{R}^n \) is offset
Linear quadratic stochastic control problem

- stage cost $g_t(x_t, u_t, w_t)$ is convex quadratic in $(x_t, u_t)$ for each $w_t$
- choose policy $u_t = \mu_t(x_t)$ to minimize objective

$$J = \mathbb{E} \left( \sum_{t=0}^{T-1} g_t(x_t, u_t, w_t) + g_T(x_T) \right)$$
Dynamic programming

- set $v_T(x) = g_T(x)$

- for $t = T - 1, \ldots, 0$,

  $$
  \mu_t(x) \in \arg\min_u \mathbb{E} \left( g_t(x, u, w_t) + v_{t+1}(f_t(x, u, w_t)) \right)
  $$

  $$
  v_t(x) = \mathbb{E} \left( g_t(x, \mu_t(x), w_t) + v_{t+1}(f_t(x, \mu_t(x), w_t)) \right)
  $$

- all $v_t$ are convex quadratic, and all $\mu_t$ are affine

- this gives value functions and optimal policy, \textit{explicitly}
Dynamic programming

we show $v_t$ are convex quadratic by (backward) induction

- suppose $v_T, \ldots, v_{t+1}$ are convex quadratic
- since $f_t$ is affine in $(x, u)$, $v_{t+1}(f_t(x, u, w_t))$ is convex quadratic
- so $g_t(x, u, w_t) + v_{t+1}(f_t(x, u, w_t))$ is convex quadratic
- and so is its expectation over $w_t$
- partial minimization over $u$ leaves convex quadratic of $x$, which is $v_t(x)$
- argmin is affine function of $x$, so optimal policy is affine
Linear equality constraints

- can add (deterministic) linear equality constraints on $x_t, u_t$ into $g_t, g_T$:

$$g_t(x, u, w) = g_t^{\text{quad}}(x, u, w) + \begin{cases} 0 & \text{if } F_t x + G_t u = h_t \\ \infty & \text{otherwise} \end{cases}$$

- everything still works:
  - $v_t$ is convex quadratic, possibly with equality constraints
  - $\mu_t$ is affine

- reason: minimizing a convex quadratic over some variables, subject to equality constraints, yields a convex quadratic in remaining variables