EE 364B: Wind Farm Layout Optimization via Sequential Convex Programming

Jinkyoo Park

1 Introduction

In a wind farm, the wakes formed by upstream wind turbines decrease the power outputs of downstream wind turbines by reducing wind speed. This wake interference significantly lowers the power production of a wind farm, especially that of a large-scale wind farm. In this paper, we express wind farm power as a differentiable function of location variables. We then apply a sequential convex programming (SCP) to optimize the locations of a large number of wind turbines to maximize the wind farm power.

2 Background

2.1 Wake model

The power output of a wind turbine is mainly determined by the wind speed profile over the rotor. We first describe how a wind turbine affects the wind speed profile. A wake model expresses the reduced wind speed at the downstream-wake distance $d$ and the radial-wake distance $r$ in the wake formed by the upstream wind turbine (Figure 1) as [Jen83]

$$u(d, r) = U(1 - \delta u(d, r)),$$

where $U$ is the undisturbed wind speed, and $\delta u(d, r)$ is the wind speed deficit factor, a ratio of wind speed reduction, due to the wake formed by the upstream wind turbine. The wind speed deficit term $\delta u(d, r) < 1$ has been expressed by a step-wise function that represent the wind speed inside the wake is constant [Jen83]. As a result, the wind farm power function constructed upon this wake model becomes non smooth, which precludes the application of a gradient based optimization algorithm to wind turbine layout optimization problem. As alternatives, search based optimization algorithms, such as Genetic Algorithm and Simulated Annealing, have been exclusively applied to this problem. However, these methods are not well scalable to a large scale wind farm composed of several hundreds of wind turbines.

Because wind speed profile have been observed to follow an inverted Gaussian function shape [Bar11], a more realistic wind speed profile can be constructed by expressing $\delta u(d, r)$
as a continuous function as

\[
\delta u(d, r) = 2\alpha \left( \frac{R}{R + \kappa d} \right)^2 \exp \left( - \left( \frac{r}{R + \kappa d} \right)^2 \right),
\]

where \( R \) is the radius of a wind turbine rotor and \((R + \kappa d)\) is the normalizing term that increases with the downstream distance \( d \). A wake expansion coefficient \( \kappa \), determined from wind speed measurement data, control the variation in the wake width with the downstream distance \( d \). A term \( \alpha \) is induction factor, which can be adjusted by the blade angle and the rotational rotor speed. The deficit factor \( \delta u(d, r) \) decreases with \( d \) and \( r \), thus successfully capturing the recovering of wind speed with the propagation of wake, as shown in Figure 1. The use of the continuous wake model allows the wind farm power to be expressed as a smooth function that can be optimized based on mathematical optimization algorithms.

### 2.2 Wind farm power model

Given a certain wind direction \( \theta_W \), the overall wind farm power will be expressed as a function of location vector \( l = (l_1^T, \ldots, l_n^T)^T \), where \( l_i = (x_i, y_i)^T \) with \( x_i \) and \( y_i \) be the coordinates of wind turbine \( i \). As shown in Figure 2, given the wind direction \( \theta_W \), we define \( d_{ij} \) as the downstream and \( r_{ij} \) as the radial wake inter distances between the hubs of the wind turbines \( i \) and \( j \), which can be expressed as

\[
d_{ij} = \|l_i - l_j\|_2 \cos(|\theta_{ij} - \theta_W|)
\]

\[
r_{ij} = \|l_i - l_j\|_2 \sin(|\theta_{ij} - \theta_W|)
\]

where \( \theta_{ij} = \sec \left( \frac{l_i \cdot l_j}{\|l_i\| \|l_j\|} \right) \) is the angle between the two wind turbines, \( i \) and \( j \).

The wake inter distances, \( d_{ij} \) and \( r_{ij} \), completely determine the wind speed profile on the rotor of the downstream wind turbine (Figure 3), which can be used to compute the power.

**Figure 1:** Continuous wake model

**Figure 2:** Definition of inter wake distances
output of the wind turbine. Based on the conservation momentum, the averaged wind speed \( \bar{u}_{ij} \) on the rotor of wind turbine \( i \) (whose rotor area is \( \pi R^2 \)) due to the wake by wind turbine \( j \) can be computed as

\[
\bar{u}_{ij}(l_i, l_j, \theta_W) = \frac{1}{\pi R^2} \int_{A_{ij}} u(d, r) dA = \int_{A_{ij}} U(1 - \delta u(d, r)) dA_{ij}
\]

where \( A_{ij} \), determined by \( d_{ij} \) and \( r_{ij} \), defines the region on the rotor of the downstream wind turbine \( i \) upon which the wind flow energy is integrated. Then, the averaged deficit factor \( \delta \bar{u}_{ij} \) can be calculated from \( \bar{u}_{ij} = U(1 - \delta \bar{u}_{ij}) \), which will be used to aggregate the influences of multiple wakes.

\[ \delta \bar{u}_i(l, \theta_W) = \sqrt{\sum_{j=1, j \neq i}^{N} (\delta \bar{u}_{ij}(l_i, l_j, \theta_W))^2}. \]

The averaged wind speed for wind turbine \( i \) influenced by wakes by all the wind turbines in a wind farm then can be expressed as

\[ \bar{u}_i(l, \theta_W) = U(1 - \delta \bar{u}_i(l, \theta_W)). \]

Accordingly, the power of wind turbine \( i \) can be expressed as a (smooth) function of the location vector \( l \) and the wind direction \( \theta_W \) as

\[
P_i(l, \theta_W) = \left( \frac{1}{2} \rho A C_P \left( \bar{u}_i(l, \theta_W) \right) \right)^3 = \left( \frac{1}{2} \rho A C_P \left( U \left( 1 - \sqrt{\sum_{j=1, j \neq i}^{N} (\delta \bar{u}_{ij}(l_i, l_j, \theta_W))^2} \right) \right)^3, \]

where \( \rho \) is the air density and \( C_P \) is the power coefficient representing the efficiency of the energy conversion from wind flow to mechanical energy.
3 Optimal wind farm layout problem

The optimal wind farm layout problem is to determine the locations of wind turbines that maximize the expected wind farm power production, which can be formulated as

\[
\begin{align*}
\text{maximize} & \quad f(l) := \mathbb{E} \left[ \sum_{i=1}^{N} P_i(l, \theta_W) \right] \\
\text{subject to} & \quad \|l_i - l_j\|_2 \geq 4D \quad \text{for} \quad i, j = 1, \ldots, N, i \neq j
\end{align*}
\]

(1)

where the expectation is taken over the distribution of the wind speed \( \theta_W \). The expected wind farm power function can be approximated as a sum over a discrete sample with the wind direction probability mass function \( \text{Pr}(\theta_W, k) \), i.e.,

\[
f(l) = \sum_{k=1}^{K} \text{Pr}(\theta_W, k) \left( \sum_{i=1}^{N} P_i(l, \theta_W, k) \right).
\]

The optimization variable \( l = (l_1^T, \ldots, l_n^T)^T \) with \( l_i = (x_i, y_i)^T \) should satisfy the two constraints. First, the inter-distance between every pair or two wind turbines should be larger than a certain limit (we set four times the rotor diameter \( D \)) for the safe operation. Second, each wind turbine is allowed to be located in a certain region, which is incorporated in the location limit vectors \( \underline{l} \) and \( \bar{l} \). The various shapes of wind farm sites can be imposed by manipulating individual components in the vectors \( \underline{l} \) and \( \bar{l} \).

4 Sequential convex programming

The derived expected wind farm power function \( f(l) \) is not concave and the constraints on the minimum inter distance is not convex, which makes it difficult to apply conventional convex programming to (1). To solve Eq. (1), we apply sequential convex programming (SCP) to solve the approximated convex programming iteratively until it reaches a local optimum. Using the analytically evaluated gradient \( \nabla f(l^{(k)}) \), and the approximated Hessian \( B^{(k)} \) based on the damped BFGS update [NW00] at the current estimate \( l^{(k)} \), the SCP approximates (1) into

\[
\begin{align*}
\text{minimize} & \quad f(l^{(k)}) + \nabla f(l^{(k)})^T (l - l^{(k)}) + \frac{1}{2} (l - l^{(k)})^T B^{(k)} (l - l^{(k)}) \\
\text{subject to} & \quad (l_i^{(k)} - l_j^{(k)})^T (l_i - l_j) \geq 4D \|l_i^{(k)} - l_j^{(k)}\|_2 \quad \text{for} \quad i, j = 1, \ldots, N, i \neq j
\end{align*}
\]

(2)

\[ l \leq l \leq \bar{l}, \quad l \in \mathcal{T}^{(k)}, \]
which can be efficiently solved by CVX [GB13]. Note that the objective function is approximated as a quadratic function and the minimum inter-distance constrains are linearized (Figure 5). The solution of (2) will be used as the next iteration point \(l^{(k+1)}\) providing that the ratio between the increase in the true function value and the increase in the approximated function value is larger than a certain limit. The constrain \(l \in T^{(k)}\), referred to as a trust region, is additionally imposed to guarantee that the approximated function is close to the true function; the approximation is close to the true objective function only around the current estimate \(l^{(k)}\). Based on the solution of (2), the trust region \(T^{(k)} = \{l||l - l^{(k)}| < \rho^{(k)}\}\) is also adjusted to guarantee the convergence to a local optimum. The overall algorithm is described in Algorithm 1.

**Algorithm 1** Wind farm layout optimization using SCP

1: initialize (choose) \(l^{(0)}\)
2: initialize the trust region \(T^{(0)} = \{l||l - l^{(0)}| < \rho^{(0)}\}\)
3: while \(||l^{(k)} - l^{(k+1)}||_2 < \epsilon\) do
4:     Find the solution \(\hat{l}\) by solving Equation (2)
5:     if \(f(l^{(k)}) - f(\hat{l})\geq \alpha\) then
6:         update the solution \(l^{(k+1)} = \hat{l}\)
7:         increase the trust region \(\rho^{(k+1)} = \beta_{\text{succ}} \rho^{(k)}\)
8:     else
9:         reject the current solution \(l^{(k+1)} = l^{(k)}\)
10:        decrease trust region \(\rho^{(k+1)} = \beta_{\text{fail}} \rho^{(k)}\)
11:     end if
12:     increment \(k \leftarrow k + 1\)
13: end while

5 Numerical Simulation

To illustrate the proposed approach, we optimize the locations of 100 wind turbines on a target site (Figure 6) with the annual wind speed distribution (Figure 7).

![Figure 6](image)

(a) Target wind farm site  (b) Annual wind direction distribution

Figure 7 compares the improvement of wind farm power efficiencies, a ratio of total wind farm power to the maximum wind farm power with no wake interference, with iterations of SCP starting from 10 different initial locations. Each of initial point is randomly disturbed one from the locations determined by the traditional design approach arraying wind turbines in a staggered grid pattern. We use different strategies of imposing box constraints on wind turbines: (a) global box constraint imposing every wind turbines to be located in a big...
global box and (b) individual box constraint imposing each wind turbine to be looted in an individually allocated region.

For two cases, the wind turbine locations with the highest efficiency among 10 simulations are compared in Figure 8. In the figure, the initial wind turbine locations are represented as blue circles, while the final optimized locations are represented as red circles. The trace of dot shows how the locations change with the iteration of SCP. The blue boxes and the filed red circles represent, respectively, the box constraints and the inter-minimum distance constraints that the final locations must satisfy. It can be shown that the individual box constraints do not lower the optimized wind farm power efficiency, though this provides flexibility for configuring different wind farm shape (by staking boxes in different shapes). Furthermore, the number of constraints can be reduced from the order of $N^2$ to the order of $N$, where $N$ is the number of wind turbines.

(a) Global box constraint  
(b) Individual box constraints

Figure 7: Optimized wind farm power efficiency

Figure 8: Optimized locations for 100 wind turbines (top of the figure faces the major wind direction NW)
References


