Optical Coherence Tomography (OCT) is an interferometric medical imaging technique that acquires depth information from a biological sample. For a single point on the surface, OCT records the amount of light reflected from each depth; one line of information is called an A-scan. The amount of light reflected back is a function of a physical parameter called the attenuation coefficient, denoted by \( \mu \). The attenuation coefficient \( \mu \) varies within the sample, and so it is a function of spatial location. An image can be constructed from a collection of A-scans generated by translating the source; each A-scan forms a single column of the image. Studies have shown that quantifying \( \mu \) is relevant for many medical applications [SGR+10, BBF+12, LGI+07, CBF+10, GML+14, YWB+11, SVBL12]; the goal of this project is to extract \( \mu \) from OCT data.

We have combined the models of Vermeer et al. [VMW+13] and Faber et al. [FMAL04] into the following model for the intensity of light recorded in an A-scan as a function of depth:

\[
I(z) = \kappa \mu(z) g(z) \exp \left( -2 \int_0^z \mu(\theta) d\theta \right),
\]

where \( I \) is the signal intensity, \( z \) is the depth, \( \mu \) is the attenuation coefficient, and \( \kappa \) is a system response parameter. The function \( g \) models the axial Point Spread Function, which is an optical effect where intensity decreases as distance from the focal plane increases. It is defined as

\[
g(z) = \left( \left( \frac{z - z_0}{z_R} \right)^2 + 1 \right)^{-1/2},
\]

where \( z_0 \) is the depth of the focal plane and \( z_R \) is the Rayleigh range. Integrating both sides of (1) and discretizing yields

\[
\sum_{j=i}^{M-1} \hat{I}_j = \frac{1}{2} \Delta z \hat{\gamma}_i - \frac{1}{2} \Delta z \hat{\gamma}_M + \frac{1}{2} \sum_{j=i}^{M-1} \frac{\hat{I}_j}{\Delta z} \hat{\gamma}_j \log \left( \frac{g(z_{j+1})}{g(z_j)} \right)
\]

for \( i = 1 \ldots M - 1 \), where \( M \) is the number of pixels in an A-scan, \( \hat{I} \in \mathbb{R}^M \) is a vector of the measured signal intensity values, \( \Delta z \) is the size of a pixel, \( \hat{\gamma}_i = 1/\hat{\mu}_i \), and \( \hat{\mu}_i \) is the
attenuation coefficient corresponding to the region imaged by the \( i^{th} \) pixel. Our goal is to recover \( \hat{\mu} \in \mathbb{R}^M \) from the known values of \( \hat{I} \), and \( \Delta z \).

System (2) is an underdetermined linear system of \( M - 1 \) equations in the \( M \) unknowns \( \hat{\gamma}_1, \ldots, \hat{\gamma}_M \), and can be written as \( A_l \hat{\gamma} = b_l \) for the correct choices of \( A_l \) and \( b_l \), where the subscript \( l \) denotes that this linear system corresponds to the \( l^{th} \) A-scan. For relevant biological samples the attenuation coefficient is piecewise constant in space. This a priori knowledge is used to rectify the system’s instability by formulating the following convex optimization problem with Total Variation regularization [ROF92]:

\[
\begin{aligned}
\text{minimize} & \quad \frac{1}{2} \| A \gamma - b \|_2^2 + \eta \| D \gamma \|_1 \\
\text{subject to} & \quad \gamma \succeq 0.
\end{aligned}
\]

where \( A = \text{diag}(A_1, A_2, \ldots, A_N) \in \mathbb{R}^{MN \times MN}; \) \( N \) is the number of A-scans in the image; \( b \in \mathbb{R}^{MN} \) is a concatenation of \( b_1, \ldots, b_N \); \( \gamma \in \mathbb{R}^{MN} \) is a concatenation of \( \hat{\gamma}_1, \ldots, \hat{\gamma}_N \); \( \eta \in \mathbb{R} \) is the regularization parameter; and \( D \) is a discrete gradient operator with symmetric boundary conditions. The matrix \( A \) has the structure shown in Figure 1. The value \( \eta \) is chosen to fit the attenuation coefficient of a calibration sample. The values of \( \hat{\mu} \) are determined by inverting each element of \( \gamma \). In this inversion if \( \gamma_i = 0 \) for some \( i \) then \( \hat{\mu}_i = \infty \), meaning that no photons penetrated that depth.

This problem can be converted into a form that can be addressed by the Alternating Direction Method of Multipliers (ADMM) algorithm. This conversion starts by rewriting problem (3) as

\[
\begin{aligned}
\text{minimize} & \quad \mathbb{I}_{\mathbb{R}_+^M} (\gamma) + F(K\gamma), \\
\text{subject to} & \quad \gamma \succeq 0.
\end{aligned}
\]

where \( \mathbb{I}_{\mathbb{R}_+^M} \) is the indicator function for the nonnegative orthant. Additionally,

\[
K = \begin{bmatrix}
A \\
D_z \\
D_z
\end{bmatrix}, \quad F \left( \begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} \right) = \frac{1}{2} \| y_1 - b \|_2^2 + \eta \| y_2 \|_1 + \eta \| y_3 \|_1.
\]
The matrices $D_z$ and $D_x$ are the discrete gradient operators in the depth and horizontal directions, respectively. By introducing the duplicate variable $u$ problem (4) is converted into

$$\begin{align*}
\text{minimize} & \quad \mathbb{I}_{\mathbb{R}^n}(u) + F(y) \\
\text{subject to} & \quad \gamma = u \\
& \quad K\gamma = y.
\end{align*}$$

(5)

The ADMM iterations for this problem are detailed in Algorithm 1.

**Algorithm 1: ADMM for Attenuation Coefficient Extraction**

1. Initialize $\gamma^1$, $u^1$, $y^1$ and $\lambda_1^1$ and $\lambda_2^1$ to 0.
2. for $k = 2, 3, \ldots$ do
3. \quad $\gamma^{k+1} = (I + K^TK)^{-1} \left( u^k + K^Ty^k - \frac{1}{\rho} \lambda_1^k - \frac{1}{\rho} K^T\lambda_2^k \right)$
4. \quad $y^{k+1} = \text{prox}_{(1/\rho)F} \left( K\gamma^{k+1} + \frac{1}{\rho}\lambda_2^k \right)$
5. \quad $u^{k+1} = \text{proj}_{\mathbb{R}^n} \left( \gamma^{k+1} + \frac{1}{\rho}\lambda_1^k \right)$
6. \quad $\lambda_1^{k+1} = \lambda_1^k + \rho \left( \gamma^{k+1} - u^{k+1} \right)$
7. \quad $\lambda_2^{k+1} = \lambda_2^k + \rho \left( K\gamma^{k+1} - y^{k+1} \right)$
8. end

By the separable sum rule of proximal operators, the proximal operator of $F$ is

$$\text{prox}_{(1/\rho)F}(x) = \begin{bmatrix}
\text{prox}_{(1/\rho)F_1}(x_1) \\
\text{prox}_{(1/\rho)F_2}(x_2) \\
\text{prox}_{(1/\rho)F_3}(x_3)
\end{bmatrix},$$

where $F_1(x_1) = (1/2)\|x_1 - b\|_2^2$, $F_2(x_2) = \eta\|x_2\|_1$, and $F_3 = F_2$. The proximal mapping operators for these functions are [PB14]

$$\text{prox}_{(1/\rho)F_1}(x_1) = \frac{\rho x_1 + b}{\rho + 1},$$

$$(\text{prox}_{(1/\rho)F_2}(x_2))_i = \begin{cases}
    x_{2,i} - 1/\rho & \text{if } x_{2,i} \geq 1/\rho \\
    0 & \text{if } |x_{2,i}| < 1/\rho \\
    x_{2,i} + 1/\rho & \text{if } x_{2,i} < -1/\rho
\end{cases}.$$

When implementing Algorithm 1, we did not create the $A, D, \text{ or } K$ matrices. Rather, we took advantage of the scientific programming interface. The matrices $A$ and $A^T$ were implemented as vectorized operations. The matrices $D_z$, $D_x$, and their transposes were implemented as shift and subtract operations. For step 3 of Algorithm 1, we used the Conjugate Gradient method. To better condition $I + K^TK$, $I$ was scaled to have values below 10. The warm start technique was employed for an improved rate of convergence of the Conjugate Gradient method. That is, the result of Conjugate Gradient from the
previous ADMM iteration was used as an initial guess for the Conjugate Gradient method of the current ADMM iteration.

Results of applying Algorithm 1 in a one-dimensional simulation are shown in Figure 2. This figure shows that the ADMM results compare favorably with truth until approximately 1.2mm of depth, after which the signal-to-noise ratio is too low to attain a good estimate of the attenuation coefficient. The figure also shows that in one-dimension the results of Algorithm 1 compare well to results achieved using CVX.

In addition to simulation, the algorithm was evaluated using real data. Test samples included: a layered phantom, and the retina of a pig. All data was collected using a Thorlabs TELESTO 1325nm OCT system equipped with a LSM03 Thorlabs lateral scanning lens with an apparent Rayleigh range of 211.81µm.

The phantom, a synthetic test object, was made with Titanium Dioxide (TiO$_2$) dissolved in Polydimethylsiloxane (PDMS) [SLK+14, LSK+14]. The attenuation coefficient of each layer was controlled by the percentage of TiO$_2$ included. The TiO$_2$ clumped together in the PDMS so the attenuation coefficient varied within a layer and presented structure similar to that found in tissue. The phantom consisted of four different layers: the first and third layer have approximately the same attenuation coefficient, the second layer has the lowest, and the fourth layer has the highest attenuation coefficient. The left portion of Figure 3 shows the OCT image of the phantom. One may note that the relative trends of the attenuation coefficients between the layers are not apparent in the raw data. The right side of Figure 3 shows the reconstructed attenuation coefficient image. As can be seen, the relative trends are accurate for the first three layers. After the third layer, the signal is attenuated so much that the signal-to-noise ratio is no longer high enough to attain an accurate estimate of the attenuation coefficient.

![Figure 3](image)

**Figure 3:** (Left) OCT image of a layered phantom. (Right) Reconstructed image of the attenuation coefficient.

Finally, the algorithm was tested on an *ex-vivo* pig’s retina. A healthy retina was harvested from a pig euthanized as part of an unrelated study. The pig was a 10-month old female Yorkshire, weighed about 50 kg, and was in good overall body condition. The result can be seen in Figure 4. Structure in the image is preserved. Again, the performance of the algorithm degrades with depth as the signal-to-noise ratio decreases.
In conclusion, we discovered an underdetermined linear system relating the measured OCT intensities to the attenuation coefficients. This system was used to formulate a convex optimization problem with Total Variation regularization to extract the attenuation coefficients from the raw data. ADMM was used to solve this convex optimization problem and extract the attenuation coefficient from raw OCT data. The algorithm was verified using simulation and data from a layered phantom; it performed well near the surface, and degraded with depth as the signal-to-noise ratio decreased. Finally, we demonstrated our algorithm on data from an ex-vivo pig’s retina.

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