Capacity Control via Convex Optimization

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EE364b: Convex Optimization II Class Project

Problem
Capacity control is a very important problem in civil infrastructure operation, but no general model exists for it. In this project we
• Formulate the real-time capacity control problem of a single facility
• Perform dynamic programming to search for the optimal solution
• Implement piecewise linear regression to update the objective function, and SCP to update the policy
• Extend the model to the capacity control problem in a network

Model
We formulate the capacity control of a single facility such that the total operation cost is minimized while the stochastic customer demands are satisfied.

\[
\begin{align*}
\text{min} & \quad \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=0}^{T} a^g(s_t, u_t, w_t) \right] \\
\text{subject to} & \quad s_{t+1} = s_t + u_t, \quad t = 0, \ldots, T-1 \\
& \quad 0 \leq s_t \leq S_{\text{max}}, \quad t = 0, \ldots, T-1 \\
& \quad U_{\text{min}} \leq u_t \leq U_{\text{max}}, \quad t = 0, \ldots, T-1
\end{align*}
\]

- \( s_t \) is the capacity at time \( t \), with the maximum \( S_{\text{max}} \).
- \( u_t \) is the control on the capacity at \( t \), with the technical limitation \( u_t \in [U_{\text{min}}, U_{\text{max}}] \).
- \( F \) is the planning horizon. \( T \to \infty \) to achieve sustainable controls.
- \( g \) is the stage cost. At each time \( t \), it is defined as

\[ g(s_t, u_t, w_t) = a(u_t) + b_t + p(s_t + u_t - w_t) \cdot \]

- \( a, b, p \in \mathbb{R} \) are the cost of capacity expansion, the cost of maintenance, and the penalty for unsatisfied demand. We assume \( p > a + b \), since otherwise the problem is trivial.
- \( w_t \) is the i.i.d stochastic demand at \( t \), sampled from a known probability distribution \( F \) that is independent of \( s_t \) or \( u_t \).
- \( \alpha \in (0, 1) \) is the discount factor.
- \( S = [0, S_{\text{max}}] \) is the state space, \( U(s) = [\text{max}(U_{\text{min}}, -s), \text{min}(U_{\text{max}}, S_{\text{max}} - s)] \) is the control space at state \( s \)

This problem is convex but cannot be directly solved since stochastic demand is involved and the time horizon is infinite. Possible approach is to find the optimal stationary policy \( \mu^* : S \to U(S) \) recursively.

Dynamic Programming Algorithm
We define the optimal value of the objective function as \( J^* : S \to \mathbb{R} \), and the DP operator \( F \) such that:

\[
F(s) = \min_{u(\cdot)} \mathbb{E} \left[ g(s, u, w) + \alpha J(s + u) \right] = \min_{u(\cdot)} \mathbb{E} \left[ g(s) + (b - a)s \right],
\]

\[
G(s) = ay + Ep(y - w) - \alpha J(0) = \|s + U_{\text{min}}\|, \min(\|U_{\text{max}} - s\|, S_{\text{max}}).
\]

We can prove \( F \) is monotone and contractive, with the Lipschitz constant \( \alpha \). Then for any bounded function \( J : S \to \mathbb{R}, J^*(s) = \lim_{k \to \infty}(F^k)(J)(s) \), for all \( s \in S \). Thus we
1. Update \( J^* = FJ \) until \( \|J - J^*\| \leq \epsilon \), to get \( J^* \) and \( G^* \).
2. For any \( s_k \), compute the optimal control \( u_k^* = \arg\min_u G^*(y) - G(y) - s_k \).

Piecewise Linear Regression and SCP
It is difficult to update \( J^* \), since \( S \) is uncountable. Fortunately, we know \( J^* \) and \( G^* \) are piecewise linear functions. So at each iteration in DP, we
1. Discretize \( S \) to get \( k \)-dimensional space \( S = \{0 \ldots \Delta_s \} : S_{\text{max}} \)
2. For each \( s \in \hat{S} \), compute optimal solution \( y^*(s) \) of \( G(s) \), and estimate \( J^\sim(s) = G(y^*(s)) + (b - a)s \)
3. Create \( K \times K \) basis matrix \( B \) such that each row is \( B(s) = [1, s, (s - \Delta_s), \ldots, (s - (K-2)\Delta_s), \ldots, (s - K\Delta_s)] \), estimate \( \theta = (B^T B)^{-1} B^T J^\sim \)
4. For any \( s' \in \hat{S} \), estimate \( J^\sim(s') = \theta^T \tilde{B}(s') \theta \)

However, by this regression, \( J^\sim \) is no more convex, but can be written as

\[ J^\sim(s) = f(s) - g(s), \text{ where} \]

\[ f(s) = \sum_{i=1}^{K} B(s)(\theta_i), g(s) = \sum_{i=1}^{K} B(s)(\theta_i) \cdot \]

Therefore we should apply SCP to obtain the optimal solution of \( G \). We start with \( y^{(0)} = S_{\text{max}}/2 \) and at each iteration \( k \), we minimize

\[ G(y) = ay + Ep(y - w) - f(y) - g(y - k) - \nabla g(y(k))T(y - g(k)) \]

Capacity Control in Network
We can easily extend our model to a facility network \( \mathcal{N} \) by using \( s_j, u_j, w_j \) as vectors, \( g = \sum_{j \in \mathcal{N}} g_j \) and adding network constraint \( s_j \in C_j \). When \( |\mathcal{N}| \) is large, we should use ADMM to find the solution locally.

Numerical examples
Consider a numerical example of single facility with \( S_{\text{max}} = 1, -U_{\text{min}} = U_{\text{max}} = 0.1, a = 1, b = 0.8, p = 2, \alpha = 0.9, w \in \{0.05, 0.25, 0.5, 0.75, 1\} \) with equal probabilities, \( S = 0 \ldots 0.05 \). We also consider a 4-facility network example, with the flow constraint \( B_{ij} \leq c, B = [1, 0, 1, 0, 1, 0, 1, 0], c = [3, 3] \).

Results
We verify that \( J^* \) converge in both single and network cases.

![Convergence of \( J^* \)](image1)

We plot \( \mu \) and \( J \) over \( S \) at some typical iterations in the single facility case. We find that the policy converges much faster than the total cost does. Also, we verify that the regression of \( J^* \) may not be convex.

![Graphs showing \( J \) and \( \mu \)](image2)

We demonstrate that our policy is better than a greedy policy \( (\alpha = 0) \) by 10 sample trajectories starting at \( s_0 = 0.5 \).

![Comparison to greedy policy](image3)

Conclusions
We propose a new model for real-time capacity control for both a single facility and a infrastructure network. We verify the convergence to the optimal solution via dynamic programming algorithm. We introduce machine learning (regression) to overcome the difficulty of updating continues cost function in DP, and use SCP to find solution of non-convex regressed function. We use numerical experiments to demonstrate the convergence and optimum of our approach.