Subsurface Detection with Convex Optimization

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EE364b: Convex Optimization II Class Project

Introduction

- non-invasive localization of subsurface, sub-wavelength, conductivity perturbations based on external electromagnetic measurements
- apply convex heuristics to a non-convex imaging problem: sequential convex programming and alternating direction method of multipliers (ADMM)

Model

- plane waves at multiple frequencies and incidence angles illuminate the imaging domain
- a perturbation, \( \sigma_u \), is buried under the surface in the region \( \Omega^1 \)
- measurements of the electric field, \( \hat{u}_j \in \Omega^2 \), are made near the surface
- all fields satisfy the Helmholtz equation with a source \( f \):
  \[
  (\nabla^2 + \omega^2 \mu + i \omega \epsilon) \hat{u} = f
  \]

Notation

Sets

- \( \Omega \) = \( 100 \times 100 \) pixel modeling domain, \( (\Delta x = \Delta y = 5 \text{m}) \)
- \( \Omega^1 \) = \( 25 \times 40 \) pixel bounding region of scattering object
- \( \Omega^2 \) = set of receiver measurements
- \( j \) = index over frequencies, illuminations

Variables

- \( u \in \mathbb{C}^{\Omega^2} \) = field measurements
- \( \hat{u} \in \mathbb{C}^{\Omega^1} \) = scattered electric field according to model in \( \Omega \)
- \( b \in \mathbb{R}^{\Omega^1} \) = background electric field according to model in \( \Omega \)
- \( p \in \mathbb{R}^{\Omega^1} \) = material perturbation vector in \( \Omega^2 \), difference \( \sigma_u - \sigma_b \)

Operators

- \( A \) = Helmholtz operator for background profile: \( \nabla^2 + k^2_0 \)
- \( \mathcal{M}_u \) = measurement operator; selects \( \Omega^2 \) from \( \Omega \)
- \( \mathcal{M}_D \) = perturbation domain operator; selects pixels \( \Omega^1 \) from \( \Omega \)

Problem Description

- want to find optimal \( u_j \) and \( p \)
  \[
  \text{minimize } \sum_j \| u_j - \mathcal{M}_u \hat{u}_j \|^2 \\
  \text{subject to } A u_j + (u_j + b_j) \circ \mathcal{M}_D(p) = 0 \quad j = 1, \ldots, N \\
  p \geq 0.
  \]
- \( \circ \) is the Hadamard, or entry-wise, product of the two vectors
- biconvex problem; if \( p \) is held constant, the problem is convex in \( u \); if \( u \) is constant, the problem is convex in \( p \)
- Helmholtz constraint contains a single non-convex term
  \[
  Au + (b \circ \mathcal{M}_D(p) + u \circ \mathcal{M}_D(p)) = 0
  \]

Sequential Born Approximation

- when the scattered wave is small with respect to the incident wave, i.e. small perturbation, the Born approximation works
- linearize the non-convex term
  \[
  (\delta u_j, \delta p_j) = \min \| u_j - \mathcal{M}_u (u_j + \delta u_j) \|^2 \\
  \text{subject to } A (u_j + \delta u_j) + \delta u_j \circ \mathcal{M}_D(p) + (u_j + b_j) \circ \mathcal{M}_D(p) = 0 \\
  \| \delta p_j \|_\infty \leq \gamma \\
  \delta p_j \geq 0.
  \]
- collect, average, and update: \( p^{k+1} = p^k + \frac{1}{N} \sum_{j=1}^N \delta p_j \)
- \( \gamma = 0.005 \) works for this problem.

ADMM

- introduce the variable \( X_j = p \circ \mathcal{M}_D(u_j + b_j) \)
- \( p \geq 120 \) works well, acts as regularization
- exactly solve at each illumination, \( j \):
  \[
  \begin{align*}
  (u_j^{k+1}, X_j^{k+1}) = \arg\min_{u_j, X_j} & \| u_j - \mathcal{M}_u u_j \|^2 + \\
  & \frac{\varsigma}{2} \| X_j - F_j \circ \mathcal{M}_D(u_j + b_j) \|^2 \\
  \text{subject to } & A u_j + \mathcal{M}_D X_j = 0 \\
  \end{align*}
  \]
- collect \( X, u \) and update for \( p \) with exact solve and thresholding:
  \[
  p^{k+1} = \arg\min_{p} \sum_{j=1}^N \| X_j^{k+1} + F_j^{k+1} - p \circ \mathcal{M}_D(u_j^{k+1} + b_j) \|^2 \\
  \text{subject to } p \geq 0.
  \]
- update dual \( F_j^{k+1} = F_j^k + X_j^{k+1} - (p^{k+1} \circ \mathcal{M}_D(u_j^{k+1} + b_j)) \)

Results

- ran algorithm 4 frequencies (1kHz, 5kHz, 10kHz, 15kHz), two incidence angles \( (45^\circ, 70^\circ) \), \( \sigma_i = 0.01 \text{ (s/m)} \), \( \sigma_b = 0.04 \text{ (s/m)} \)
- 30 surface measurements
- 15 sequential Born approximations iterations takes 38 minutes (Intel i5, 2.8GHz) with CVX.
- 100 iterations of ADMM takes 6 minutes (Intel i5, 2.8GHz)

Conclusions

- ADMM and SBA give similar results, ADMM currently much faster
- add total variation regularization to ADMM algorithm, promote sharp edges
- custom interior point solver for sequential Born approximation
- noisy measurements introduced at each iteration gives similar performance