Computing the Capacity of the Degraded Relay Channel
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Problem Introduction

- The most basic model in communications consists of a single sender, receiver, and channel that may corrupt the sender’s message. A primary quantity of interest is the capacity of the channel, which measures how much information may be sent through the channel.
- The relay channel is a modification of this simple model. In the relay channel, there is a third entity (the relay) between the sender and receiver that assists the sender in transmitting information to the receiver.
- The basic model is well understood and the capacity is easy to find. The relay channel is far more difficult and currently unsolved in the general case. We present a method for numerically computing the capacity of the relay channel in one important special case.

Information Theory Background

- The standard communication model is defined by an input alphabet, \( x \in X = \{1, \ldots, n\} \), an output alphabet \( y \in Y = \{1, \ldots, m\} \), and a transition matrix \( P \in \mathbb{R}^{n \times m} \). The entry \( p_{ij} \) is equal to \( p(y = i | x = j) \).
- The mutual information \( I(X;Y) \) between two probability distributions is defined as:

\[
I(X;Y) = \sum_{y \in Y} \sum_{x \in X} p(x)p(y|x) \log \frac{p(y|x)}{p(y)}
\]

- The capacity of a channel is given by

\[
C = \max_{p(x)} I(X;Y)
\]

- Mutual information and capacity are both concave functions in \( p(x) \).

Relay Channel Description

- The relay channel contains two additional alphabets: the message seen by the relay \( y_1 \in Y_1 = \{1, \ldots, k\} \), and the message sent by the relay \( x_2 \in X_2 = \{1, \ldots, l\} \).
- The channel characteristics are now governed by the distribution \( p(y, y_1, x_1, x_2) \).
- The capacity of the general case channel is unknown. A special case is the physically degraded relay channel, for which an achievable capacity bound is known. In this scenario, \( X_1 \) and \( Y \) are conditionally independent given \( (X_2, Y_1) \). The capacity of the degraded relay channel is

\[
C = \max_{p(x_1, x_2)} \min_{p(y_1)} I(X_1;X_2, Y), I(X_1; Y_1 | X_2)
\]

Algorithm

- The optimization problem we would like to solve (with variable \( p(x_1, x_2) \)) is

\[
\begin{align*}
& \text{maximize} \quad \min_{y_1} \left\{ I(X_1; X_2, Y), I(X_1; Y_1 | X_2) \right\} \\
& \text{subject to} \quad \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p(x_1, x_2) = 1 \\
& \quad \quad \quad \quad p(x_1, x_2) \geq 0, \quad x_1 \in X_1, \quad x_2 \in X_2
\end{align*}
\]

Example Problem and Results

- Our algorithm produces an exact solution for all of the toy cases in which we know the answer.
- To overcome this, we employ a relax-and-project heuristic when optimizing \( p(x_1, x_2) \) alternatively maximizing over \( y_1 \) and \( p(x_1, x_2) \), so we attempt to approximately solve the problem by maximizing only \( I(X_1; X_2, Y) \).
- To simplify the optimization problem, we attempt to approximately solve the problem by maximizing only \( I(X_1; X_2, Y) \).
- Our method is to alternate solving the two optimization problems:

\[
\begin{align*}
& \text{maximize} \quad I(X_1; X_2, Y) \\
& \quad \text{subject to} \quad \sum_{x_2} p(x_2) = 1 \\
& \quad \quad \quad \quad p(x_2 = j) \geq 0, \quad j = 1, \ldots, m
\end{align*}
\]

where the optimization variable is the marginal distribution \( p(x_2) \), and

\[
\begin{align*}
& \text{maximize} \quad \min_{y_1} \left\{ I(X_1; X_2, Y), I(X_1; Y_1 | X_2) \right\} \\
& \quad \text{subject to} \quad \sum_{x_2} \sum_{x_1} p(x_1 | x_2) = 1 \\
& \quad \quad \quad \quad p(x_1 | x_2 = j) \geq 0, \quad j = 1, \ldots, m
\end{align*}
\]

where the optimization variable is the conditional distribution \( p(x_1 | x_2) \).

Conclusions

- By applying several heuristics, we have an approach to finding the approximate capacity of a degraded relay channel.
- We cannot guarantee that our solution achieves the maximum, but it seems to work well for the cases in which we know the answer.