# Subgradient Methods

- subgradient method and stepsize rules
- convergence results and proof
- optimal step size and alternating projections
- speeding up subgradient methods

# Subgradient method

**subgradient method** is simple algorithm to minimize nondifferentiable convex function $f$

$$x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)}$$

- $x^{(k)}$ is the $k$th iterate
- $g^{(k)}$ is **any** subgradient of $f$ at $x^{(k)}$
- $\alpha_k > 0$ is the $k$th step size

not a descent method, so we keep track of best point so far

$$f_{\rm best}^{(k)} = \min_{i=1,\ldots,k} f(x^{(i)})$$

# Step size rules

step sizes are fixed ahead of time

- *constant step size:* $\alpha_k = \alpha$ (constant)

- *constant step length:* $\alpha_k = \gamma / \|g^{(k)}\|_2$ (so $\|x^{(k+1)} - x^{(k)}\|_2 = \gamma$)

- *square summable but not summable:* step sizes satisfy

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$

- *nonsummable diminishing:* step sizes satisfy

$$\lim_{k \to \infty} \alpha_k = 0, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$

# Assumptions

- $f^\star = \inf_x f(x) > -\infty$, with $f(x^\star) = f^\star$
- $\|g\|_2 \leq G$ for all $g \in \partial f$ (equivalent to Lipschitz condition on $f$)
- $\|x^{(1)} - x^\star\|_2 \leq R$

these assumptions are stronger than needed, just to simplify proofs

# Convergence results

define $\bar{f} = \lim_{k\to\infty} f_{\rm best}^{(k)}$

- *constant step size:* $\bar{f} - f^\star \leq G^2\alpha/2$, $i.e.$, **converges to** $G^2\alpha/2$**-suboptimal** (converges to $f^\star$ if $f$ differentiable, $\alpha$ small enough)
- *constant step length:* $\bar{f} - f^\star \leq G\gamma/2$, $i.e.$, **converges to** $G\gamma/2$**-suboptimal**
- *diminishing step size rule:* $\bar{f} = f^\star$, $i.e.$, **converges**

# Convergence proof

**key quantity:** *Euclidean distance to the optimal set*, not the function value

let $x^\star$ be any minimizer of $f$

$$\begin{array}{rcl}\|x^{(k+1)} - x^\star\|_2^2 & = & \|x^{(k)} - \alpha_k g^{(k)} - x^\star\|_2^2 \\ & = & \|x^{(k)} - x^\star\|_2^2 - 2\alpha_k g^{(k)T}(x^{(k)} - x^\star) + \alpha_k^2 \|g^{(k)}\|_2^2 \\ & \leq & \|x^{(k)} - x^\star\|_2^2 - 2\alpha_k (f(x^{(k)}) - f^\star) + \alpha_k^2 \|g^{(k)}\|_2^2\end{array}$$

using $f^\star = f(x^\star) \geq f(x^{(k)}) + g^{(k)T}(x^\star - x^{(k)})$

apply recursively to get

$$
\begin{aligned}
\|x^{(k+1)} - x^\star\|_2^2 &\leq \|x^{(1)} - x^\star\|_2^2 - 2\sum_{i=1}^{k} \alpha_i (f(x^{(i)}) - f^\star) + \sum_{i=1}^{k} \alpha_i^2 \|g^{(i)}\|_2^2 \\
&\leq R^2 - 2\sum_{i=1}^{k} \alpha_i (f(x^{(i)}) - f^\star) + G^2 \sum_{i=1}^{k} \alpha_i^2
\end{aligned}
$$

now we use

$$
\sum_{i=1}^{k} \alpha_i (f(x^{(i)}) - f^\star) \geq (f_{\mathrm{best}}^{(k)} - f^\star) \left( \sum_{i=1}^{k} \alpha_i \right)
$$

to get

$$
f_{\mathrm{best}}^{(k)} - f^\star \leq \frac{R^2 + G^2 \sum_{i=1}^{k} \alpha_i^2}{2 \sum_{i=1}^{k} \alpha_i}.
$$

**constant step size:** for $\alpha_k = \alpha$ we get

$$f_{\rm best}^{(k)} - f^\star \le \frac{R^2 + G^2 k\alpha^2}{2k\alpha}$$

righthand side converges to $G^2\alpha/2$ as $k \to \infty$

**constant step length:** for $\alpha_k = \gamma/\|g^{(k)}\|_2$ we get

$$f_{\rm best}^{(k)} - f^\star \le \frac{R^2 + \sum_{i=1}^k \alpha_i^2 \|g^{(i)}\|_2^2}{2\sum_{i=1}^k \alpha_i} \le \frac{R^2 + \gamma^2 k}{2\gamma k/G},$$

righthand side converges to $G\gamma/2$ as $k \to \infty$

**square summable but not summable step sizes:**
suppose step sizes satisfy

$$\sum_{i=1}^{\infty} \alpha_k^2 < \infty, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$

then

$$f_{\rm best}^{(k)} - f^\star \leq \frac{R^2 + G^2 \sum_{i=1}^{k} \alpha_i^2}{2\sum_{i=1}^{k} \alpha_i}$$

as $k \to \infty$, numerator converges to a finite number, denominator converges to $\infty$, so $f_{\rm best}^{(k)} \to f^\star$

# Stopping criterion

- terminating when $\dfrac{R^2 + G^2 \sum_{i=1}^k \alpha_i^2}{2 \sum_{i=1}^k \alpha_i} \leq \epsilon$ is really, really, slow

- optimal choice of $\alpha_i$ to achieve $\dfrac{R^2 + G^2 \sum_{i=1}^k \alpha_i^2}{2 \sum_{i=1}^k \alpha_i} \leq \epsilon$ for smallest $k$:

$$\alpha_i = (R/G)/\sqrt{k}, \quad i = 1, \ldots, k$$

  number of steps required: $k = (RG/\epsilon)^2$

- the truth: there really isn't a good stopping criterion for the subgradient method . . .

# Example: Piecewise linear minimization

minimize $\quad f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$

to find a subgradient of $f$: find index $j$ for which

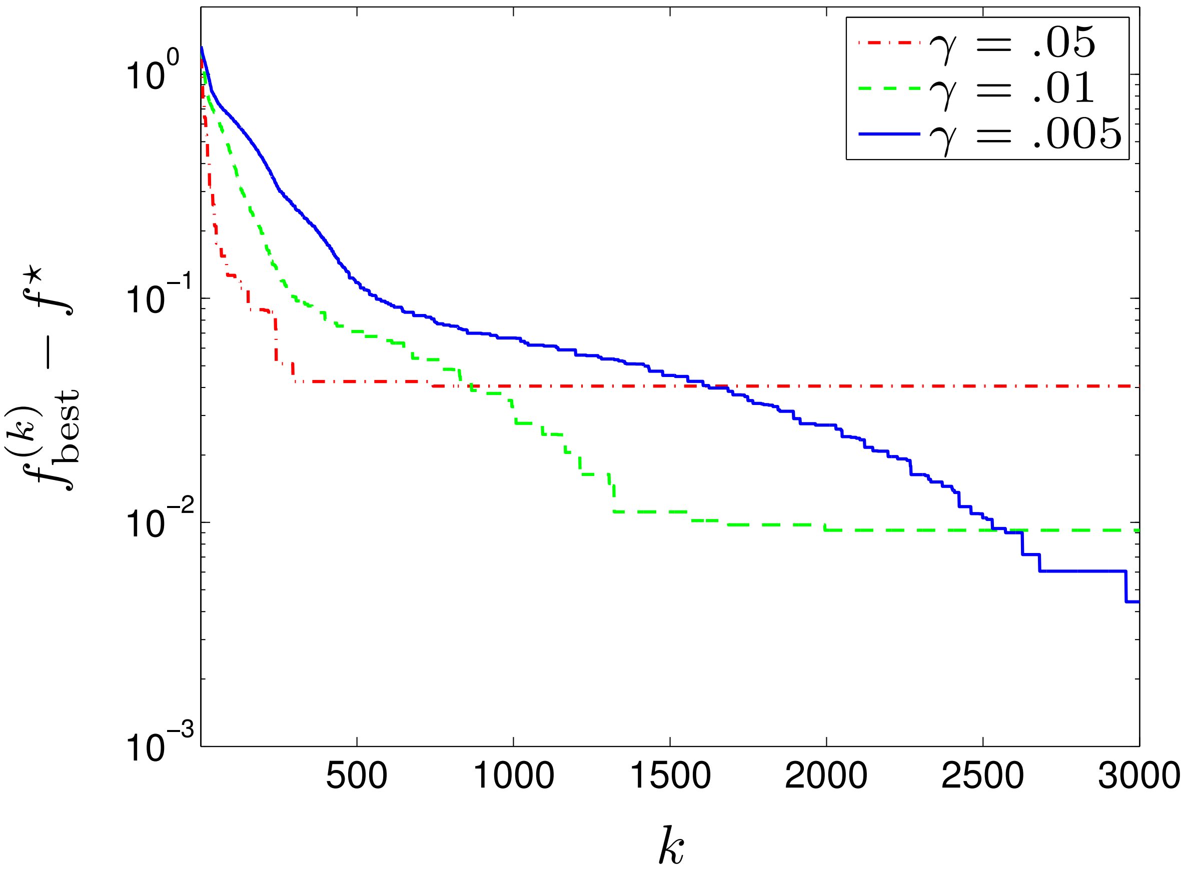
$$a_j^T x + b_j = \max_{i=1,\ldots,m}(a_i^T x + b_i)$$

and take $g = a_j$

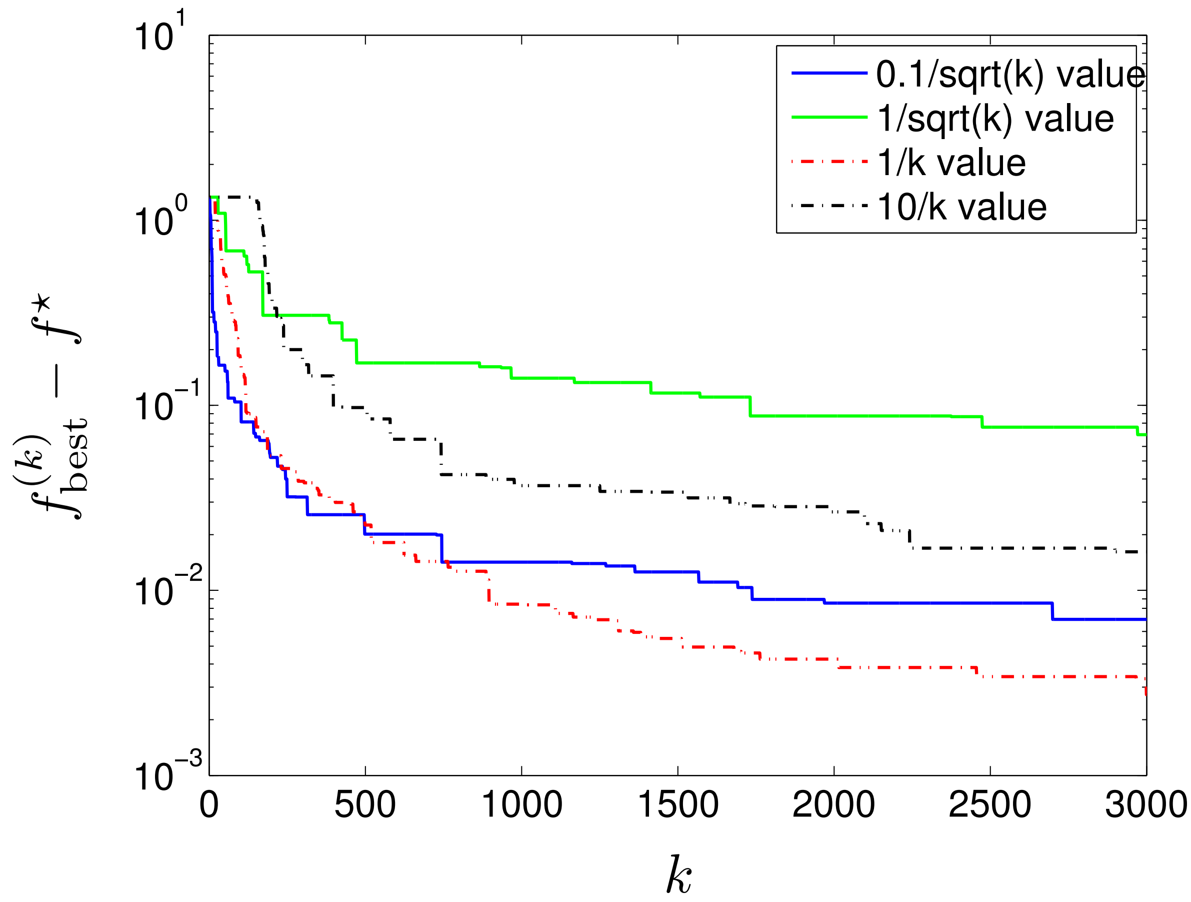
subgradient method: $x^{(k+1)} = x^{(k)} - \alpha_k a_j$

problem instance with $n = 20$ variables, $m = 100$ terms, $f^\star \approx 1.1$

$f_{\mathrm{best}}^{(k)} - f^\star$, constant step length $\gamma = 0.05, 0.01, 0.005$



diminishing step rules $\alpha_k = 0.1/\sqrt{k}$ and $\alpha_k = 1/\sqrt{k}$, square summable step size rules $\alpha_k = 1/k$ and $\alpha_k = 10/k$



# Optimal step size when $f^\star$ is known

- choice due to Polyak:
$$\alpha_k = \frac{f(x^{(k)}) - f^\star}{\|g^{(k)}\|_2^2}$$
(can also use when optimal value is estimated)

- motivation: start with basic inequality
$$\|x^{(k+1)} - x^\star\|_2^2 \leq \|x^{(k)} - x^\star\|_2^2 - 2\alpha_k(f(x^{(k)}) - f^\star) + \alpha_k^2\|g^{(k)}\|_2^2$$
and choose $\alpha_k$ to minimize righthand side

- yields

$$\|x^{(k+1)} - x^\star\|_2^2 \le \|x^{(k)} - x^\star\|_2^2 - \frac{(f(x^{(k)}) - f^\star)^2}{\|g^{(k)}\|_2^2}$$

(in particular, $\|x^{(k)} - x^\star\|_2$ decreases each step)

- applying recursively,

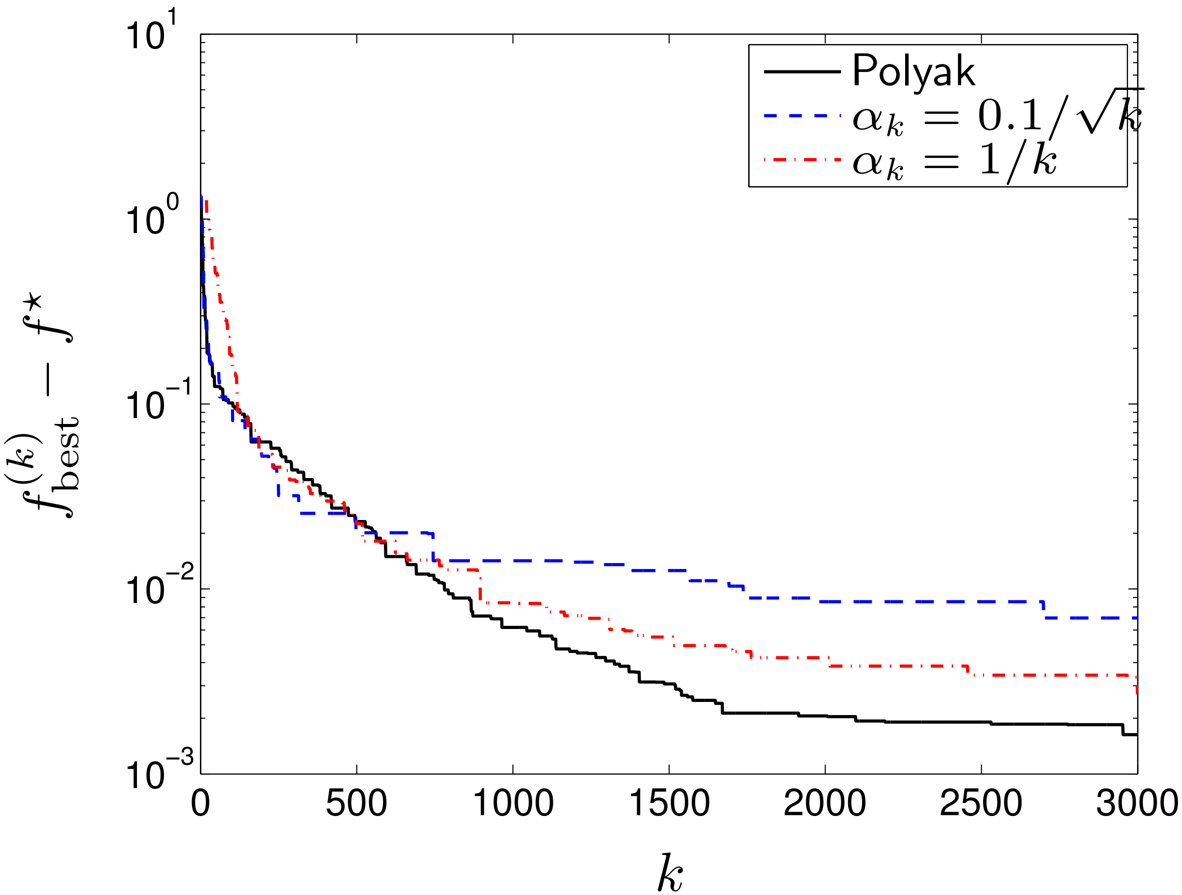
$$\sum_{i=1}^{k} \frac{(f(x^{(i)}) - f^\star)^2}{\|g^{(i)}\|_2^2} \le R^2$$

and so

$$\sum_{i=1}^{k} (f(x^{(i)}) - f^\star)^2 \le R^2 G^2$$

which proves $f(x^{(k)}) \to f^\star$

PWL example with Polyak's step size, $\alpha_k = 0.1/\sqrt{k}$, $\alpha_k = 1/k$



# Finding a point in the intersection of convex sets

$C = C_1 \cap \cdots C_m$ is nonempty, $C_1, \ldots, C_m \subseteq \mathbf{R}^n$ closed and convex

find a point in $C$ by minimizing

$$f(x) = \max\{\mathbf{dist}(x, C_1), \ldots, \mathbf{dist}(x, C_m)\}$$

with $\mathbf{dist}(x, C_j) = f(x)$, a subgradient of $f$ is

$$g = \nabla\,\mathbf{dist}(x, C_j) = \frac{x - P_{C_j}(x)}{\|x - P_{C_j}(x)\|_2}$$

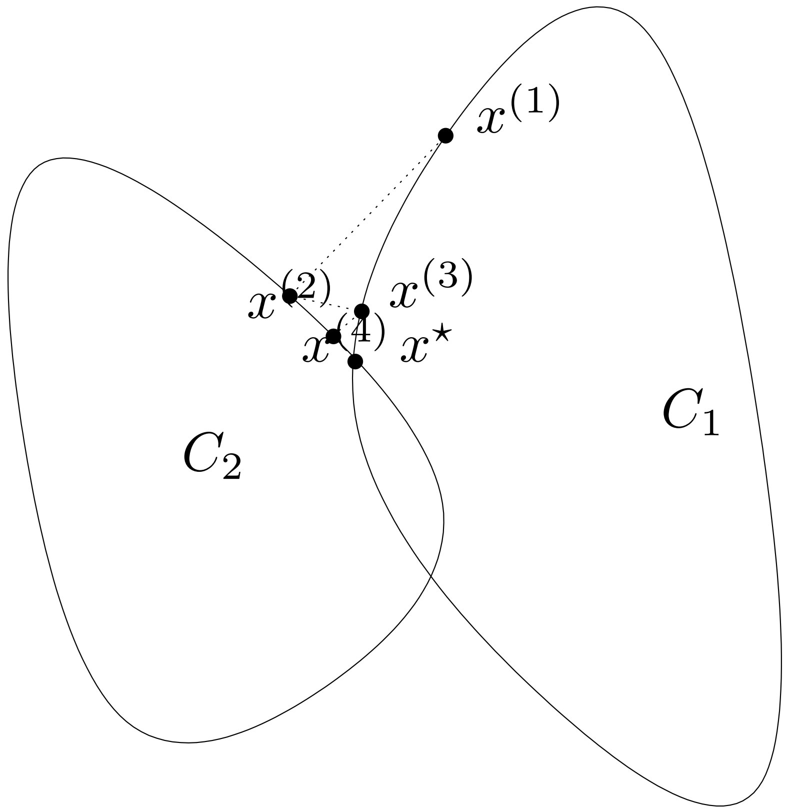
subgradient update with optimal step size:

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - \alpha_k g^{(k)} \\ &= x^{(k)} - f(x^{(k)}) \frac{x - P_{C_j}(x)}{\|x - P_{C_j}(x)\|_2} \\ &= P_{C_j}(x^{(k)}) \end{aligned}$$

- a version of the famous *alternating projections* algorithm
- at each step, project the current point onto the farthest set
- for $m = 2$ sets, projections alternate onto one set, then the other
- convergence: $\mathbf{dist}(x^{(k)}, C) \to 0$ as $k \to \infty$

# Alternating projections

first few iterations:



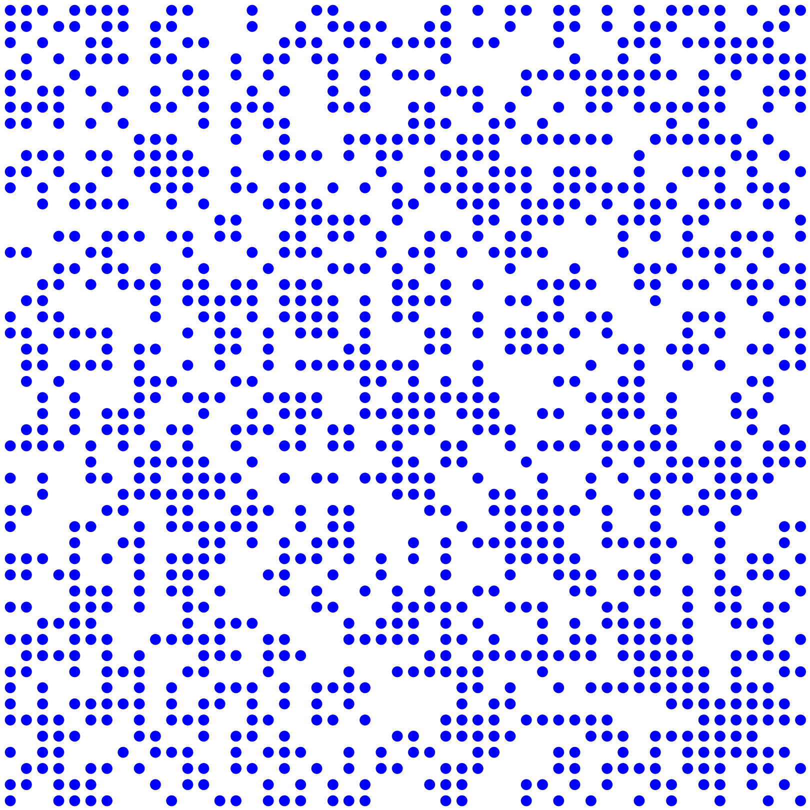

. . . $x^{(k)}$ eventually converges to a point $x^\star \in C_1 \cap C_2$

## Example: Positive semidefinite matrix completion

- some entries of matrix in $\mathbf{S}^n$ fixed; find values for others so completed matrix is PSD
- $C_1 = \mathbf{S}^n_+$, $C_2$ is (affine) set in $\mathbf{S}^n$ with specified fixed entries
- projection onto $C_1$ by eigenvalue decomposition, truncation: for $X = \sum_{i=1}^n \lambda_i q_i q_i^T$,

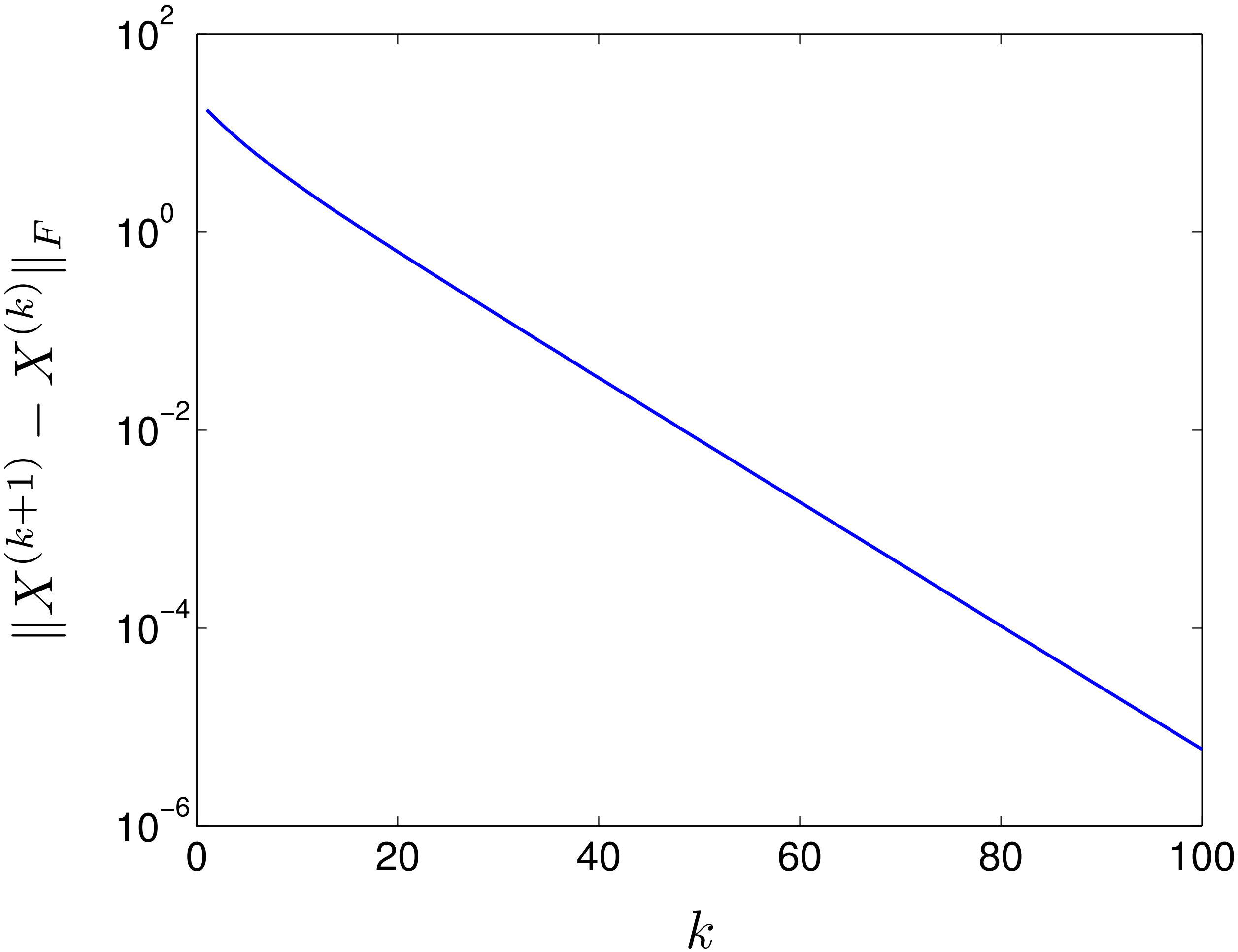
$$P_{C_1}(X) = \sum_{i=1}^n \max\{0, \lambda_i\} q_i q_i^T$$

- projection of $X$ onto $C_2$ by re-setting specified entries to fixed values specific example: $50 \times 50$ matrix missing about half of its entries



- initialize $X^{(1)}$ with unknown entries set to $0$

convergence is linear:



# Polyak step size when $f^\star$ isn't known

- use step size

$$\alpha_k = \frac{f(x^{(k)}) - f_{\rm best}^{(k)} + \gamma_k}{\|g^{(k)}\|_2^2}$$

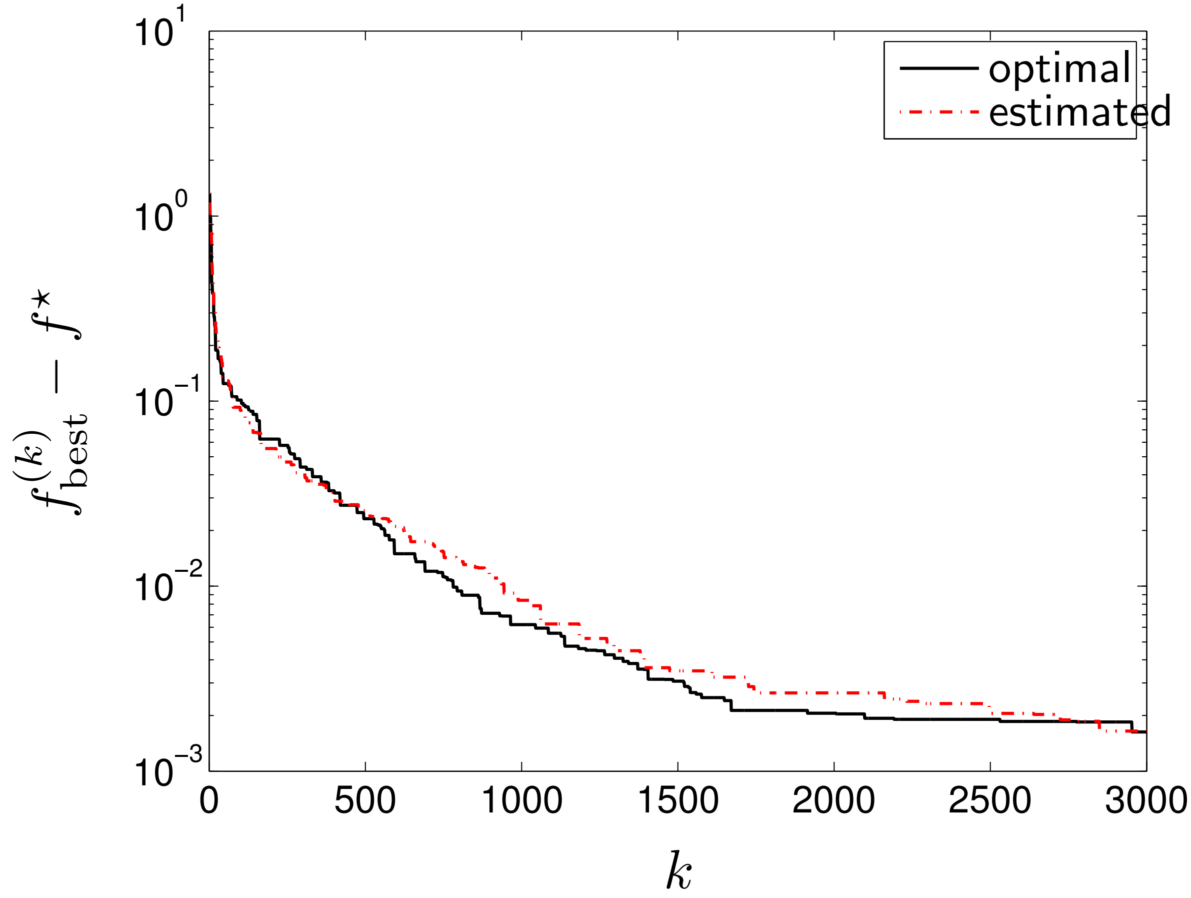
with $\sum_{k=1}^\infty \gamma_k = \infty$, $\sum_{k=1}^\infty \gamma_k^2 < \infty$

- $f_{\rm best}^{(k)} - \gamma_k$ serves as estimate of $f^\star$

- $\gamma_k$ is in scale of objective value

- can show $f_{\rm best}^{(k)} \to f^\star$

PWL example with Polyak's step size, using $f^\star$, and estimated with $\gamma_k = 10/(10+k)$



# Speeding up subgradient methods

- subgradient methods are very slow
- often convergence can be improved by keeping memory of past steps

$$x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)} + \beta_k (x^{(k)} - x^{(k-1)})$$

(heavy ball method)

**other ideas**: localization methods, conjugate directions, . . .

# A couple of speedup algorithms

$$x^{(k+1)} = x^{(k)} - \alpha_k s^{(k)}, \qquad \alpha_k = \frac{f(x^{(k)}) - f^\star}{\|s^{(k)}\|_2^2}$$

(we assume $f^\star$ is known or can be estimated)

- 'filtered' subgradient, $s^{(k)} = (1-\beta)g^{(k)} + \beta s^{(k-1)}$, where $\beta \in [0,1)$
- Camerini, Fratta, and Maffioli (1975)

$$s^{(k)} = g^{(k)} + \beta_k s^{(k-1)}, \qquad \beta_k = \max\{0, -\gamma_k (s^{(k-1)})^T g^{(k)} / \|s^{(k-1)}\|_2^2\}$$

where $\gamma_k \in [0,2)$ ($\gamma_k = 1.5$ 'recommended')

PWL example, Polyak's step, filtered subgradient, CFM step