# Semidefinite Relaxations and Applications

- Semidefinite relaxations
- Lagrangian relaxations for QCQPs
- Randomization
- Bounds on suboptimality
- Applications

# Nonconvex problems

ee364 (more or less correct) view:

- **convex** is easy
- **nonconvex** is hard(er)

we will use convex optimization to

- find bounds on optimal value by **relaxation**
- get "good enough" feasible points by **randomization**

# Basic problem: QCQPs

$$\begin{array}{ll} \text{minimize} & x^T A_0 x + b_0^T x + c_0 \\ \text{subject to} & x^T A_i x + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m. \end{array}$$

- if all $A_i$ are PSD, convex problem, use ee364
- here, we suppose at least one $A_i$ is not PSD

# Example: Boolean Least Squares

Boolean least-squares problem is to

$$\text{minimize } \|Ax - b\|_2^2 \text{ subject to } \quad x_i^2 = 1, \quad i = 1, \ldots, n$$

- basic problem in digital communications (noisy channel)
- could check all $2^n$ possible values of $x$ . . .
- an NP-hard problem, and very hard in practice
- many heuristics for approximate solution

# Example: Partitioning Problem

two-way partitioning problem (§5.1.5 in [BV04]):

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

where $W = W^T$, $W_{ii} = 0$

- feasible $x \in \{-1, 1\}$ corresponds to partitioning
- coefficients $W_{ij}$ interpreted as the cost of having the elements $i$ and $j$ in the same partition.
- the objective is to find the partition with least total cost
- classic particular instance: MAXCUT $(W_{ij} \geq 0)$

# Example: cardinality problems

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

introduce $z_i \in \{0, 1\}$, i.e. $z_i(1 - z_i) = 0$,

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & z_i - z_i^2 = 0, \;\; x_i(1 - z_i) = 0 \quad i = 1, \ldots, n \\ & x \in \mathcal{C} \end{array}$$

# Semidefinite relaxation

original QCQP

$$\begin{array}{ll} \text{minimize} & x^T A_0 x + b_0^T x + c_0 \\ \text{subject to} & x^T A_i x + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m. \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(A_0 X) + b_0^T x + c_0 \\ \text{subject to} & \mathbf{Tr}(A_i X) + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m \\ & X = xx^T \end{array}$$

change $X = xx^T$ into $X \succeq xx^T$

# Lagrangian relaxation

original QCQP

$$\begin{array}{ll}\text{minimize} & x^T A_0 x + b_0^T x + c_0 \\ \text{subject to} & x^T A_i x + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m.\end{array}$$

forming Lagrangian

$$L(x, \lambda) = x^T \Big(A_0 + \sum_{i=1}^m \lambda_i A_i\Big) x + \Big(b_0 + \sum_{i=1}^m \lambda_i b_i\Big)^T x + c_0 + \lambda^T c$$

recall that

$$\inf_x \{x^T P x + q^T x + r\} = \begin{cases} r - \frac{1}{4} q^T P^\dagger q & \text{if } P \succeq 0, \quad q \in \mathcal{R}(P) \\ -\infty & \text{otherwise} \end{cases}$$

# Lagrangian relaxation: dual

$$L(x,\lambda) = x^T\Big(A_0 + \sum_{i=1}^m \lambda_i A_i\Big)x + \Big(b_0 + \sum_{i=1}^m \lambda_i b_i\Big)^T x + c_0 + \lambda^T c$$

has (for $B = [b_1 \ \cdots b_m]^T \in \mathbf{R}^{m\times n}$)

$$\begin{aligned} g(\lambda) &= \inf_x L(x,\lambda) \\ &= -\frac{1}{4}(b_0 + B^T\lambda)^T\Big(A_0 + \sum_i \lambda_i A_i\Big)^\dagger (b_0 + B^T\lambda) + \lambda^T c + c_0 \end{aligned}$$

# Lagrangian relaxation: dual

Taking Schur complements gives dual problem

$$\begin{array}{ll} \text{maximize} & \frac{1}{4}\gamma + c^T\lambda + c_0 \\ \text{subject to} & \begin{bmatrix} (A_0 + \sum_{i=1}^m \lambda_i A_i) & (b_0 + B^T\lambda) \\ (b_0 + B^T\lambda)^T & -\gamma \end{bmatrix} \succeq 0, \\ & \lambda \succeq 0 \end{array}$$

semidefinite program in variable $\lambda \in \mathbf{R}_+^m$ and can be solved "efficiently"

# Lagrangian relaxation: Bidual

Taking dual again gives SDP

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(A_0X) + b_0^Tx + c_0 \\ \text{subject to} & \mathbf{Tr}(A_iX) + b_i^Tx + c_i \leq 0, \quad i = 1, \ldots, m \\ & \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 \end{array}$$

in variables $X \in \mathbf{S}^n$, $x \in \mathbf{R}^n$

- have recovered original SDP relaxation "automatically"
- convexification of original problem!

# Example: Partitioning

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

no need to maintain variable $x$, gives relaxation (via $X = xx^T$)

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(WX) \\ \text{subject to} & X \succeq 0, \ \ \mathbf{diag}(X) = \mathbf{1} \end{array}$$

# Feasible points?

- have lower bounds on optimal value of problem
- big question: how do we compute good feasible points?
- can we measure if our lower bound is suboptimal?

# Simplest idea: randomization

original problem

$$\begin{array}{ll} \text{minimize} & x^T A_0 x + b_0^T x + c_0 \\ \text{subject to} & x^T A_i x + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m. \end{array}$$

and relaxation

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(A_0 X) + b_0^T x + c_0 \\ \text{subject to} & \mathbf{Tr}(A_i X) + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m \\ & X - xx^T \succeq 0 \end{array}$$

- if $X, x$ solve relaxed problem, then $X - xx^T \succeq 0$ can be a covariance matrix.

# Gaussian randomization

- pick $z$ as a Gaussian variable with $z \sim \mathcal{N}(x, X - xx^T)$
- $z$ will solve the QCQP "on average" over this distribution

in other words:

$$\begin{array}{ll} \text{minimize} & \mathbf{E}[z^T A_0 z + b_0^T z + r_0] \\ \text{subject to} & \mathbf{E}[z^T A_i z + b_i^T z + c_i] \leq 0, \quad i = 1, \ldots, m \end{array}$$

a good feasible point obtained by sampling enough $z$ (often more sophisticated strategies)

# Gaussian randomization

- possible to get sharper guarantees and exactly feasible points, e.g. for MAXCUT or other boolean problems
- constraint

  $$x_i^2 = 1$$

  so just take $x_i = \mathbf{sign}(z_i)$
- for $\hat{x} = \mathbf{sign}(z_i)$, $z_i \sim \mathcal{N}(0, X)$, have

  $$\mathbf{E}[\hat{x}_i \hat{x}_j] = \frac{2}{\pi} \arcsin(X_{ij})$$

# Approximation guarantees

MAXCUT relaxation

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}(WX) \\ \text{subject to} & \mathbf{diag}(X) = \mathbf{1}, X \succeq 0 \end{array}$$

gives

$$\mathbf{E}[\hat{x}^T W \hat{x}] = \frac{2}{\pi} \mathbf{E}[W \arcsin(X)]$$

- draw a few samples $\hat{x}$, get at least that good with high probability
- optimal value of MAXCUT is between $\frac{2}{\pi} \mathbf{Tr}(W \arcsin(X))$ and $\mathbf{Tr}(WX)$.

# Better rounding (Goemans & Williamson)

suppose $W_{ij} \geq 0$, maximize

$$\sum_{ij} W_{ij}(1 - X_{ij}) \quad \text{subject to} \quad \mathbf{diag}(X) = \mathbf{1}, \; X \succeq 0$$

- sample coordinates $\hat{x}_i$ at random, get $\mathbf{Tr}(W) - \mathbf{E}[\hat{x}^T W \hat{x}] = \mathbf{Tr}(W)$, at least 50% optimal
- sample directions:

$$X_{ij} = v_i^T v_j \quad \text{with} \quad \|v_i\| = 1$$

  i.e. $X = V^T V$ by Cholesky
- draw $Z$ uniformly at random on unit sphere, set

$$\hat{x}_i = \mathbf{sign}(Z^T v_i)$$

# Better rounding (Goemans & Williamson)

expected value of cut is

$$\begin{aligned}
\mathbf{E}[W_{ij}(1-\hat{x}_i\hat{x}_j)] &= 2W_{ij}\,\mathrm{Pr}(Z \text{ separates } v_i, v_j) \\
&= 2W_{ij}\,\mathrm{Pr}(\mathbf{sign}(v_i^T Z) \neq \mathbf{sign}(v_j^T Z)) \\
&= 2W_{ij}\frac{2\theta(v_i, v_j)}{2\pi} \\
&= \frac{2}{\pi}W_{ij}\cos^{-1}(v_i^T v_j)
\end{aligned}$$

so

$$\sum_{ij}\mathbf{E}[W_{ij}(1-\hat{x}_i\hat{x}_j)] = \frac{2}{\pi}\sum_{ij}W_{ij}\cos^{-1}(X_{ij})$$

- Fact: $\cos^{-1}(t) \geq \frac{\pi}{2}\alpha(1-t)$, $\alpha \approx .87856$

# Better rounding: final bound

- expected weight from random cut generated by optimal $X$ is at least

$$\frac{2}{\pi}\sum_{ij} W_{ij}\cos^{-1}(X_{ij}) \geq \alpha \sum_{ij} W_{ij}(1-X_{ij}) = \alpha \mathsf{SDP}^*.$$

- alternatives: if $W \succeq 0$, then (Nesterov 98)

$$\mathbf{Tr}(W \arcsin(X)) \geq \mathbf{Tr}(WX)$$

so (using earlier bound)

$$\mathsf{SDP}^* \geq \mathsf{OPT} \geq \frac{2}{\pi}\mathsf{SDP}^*$$

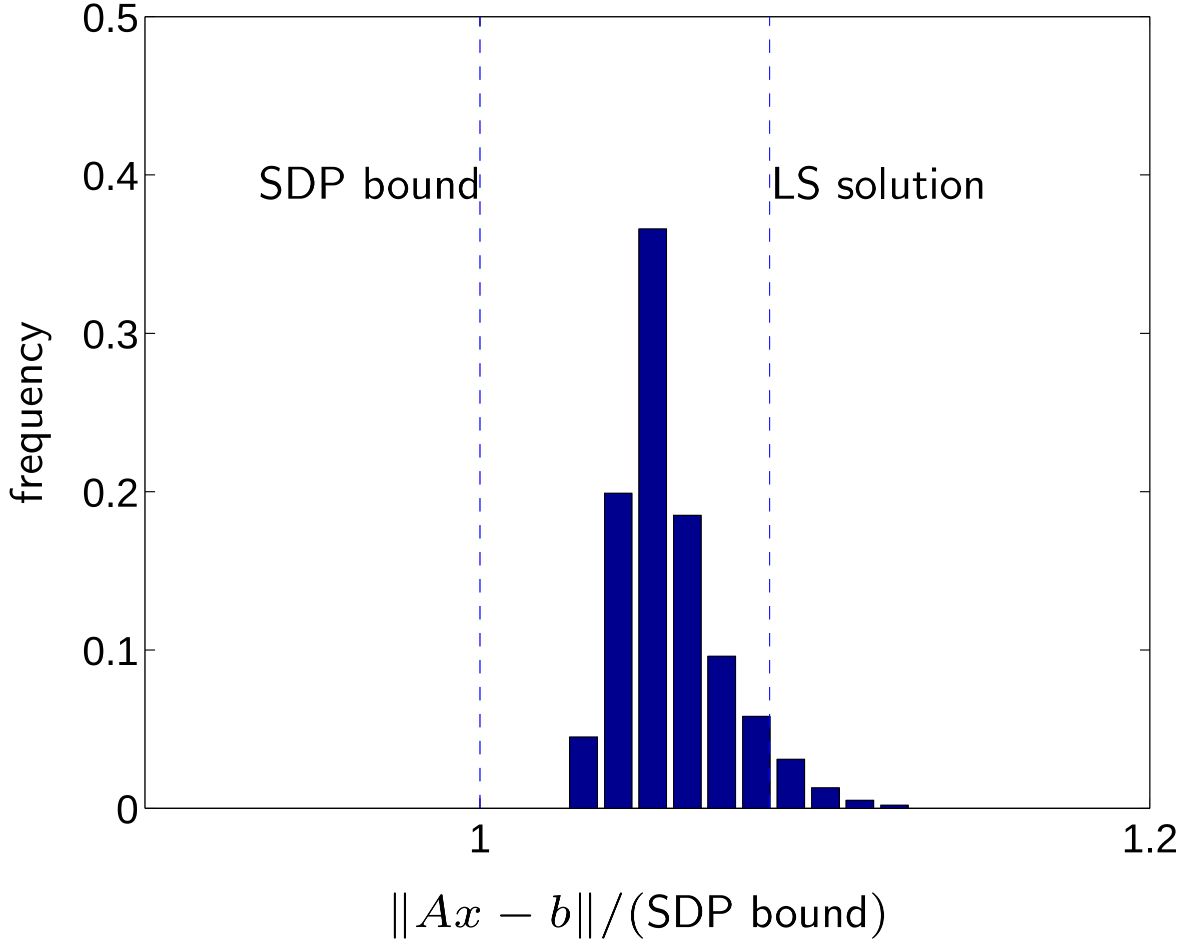
# Example: boolean least squares

- (randomly chosen) parameters $A \in \mathbf{R}^{150 \times 100}$, $b \in \mathbf{R}^{150}$
- $x \in \mathbf{R}^{100}$, so feasible set has $2^{100} \approx 10^{30}$ points

**LS approximate solution:** minimize $\|Ax - b\|$ s.t. $\|x\|_2^2 \leq n$, then round yields objective 8.7% over SDP relaxation bound

**randomized method:** (using SDP optimal distribution)

- best of 20 samples: $3.1\%$ over SDP bound
- best of 1000 samples: $2.6\%$ over SDP bound


0.5
0.4
0.3
0.2
0.1
0
frequency
SDP bound
LS solution
1
1.2
$\|Ax - b\|/(\text{SDP bound})$

# Example: partitioning problem

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

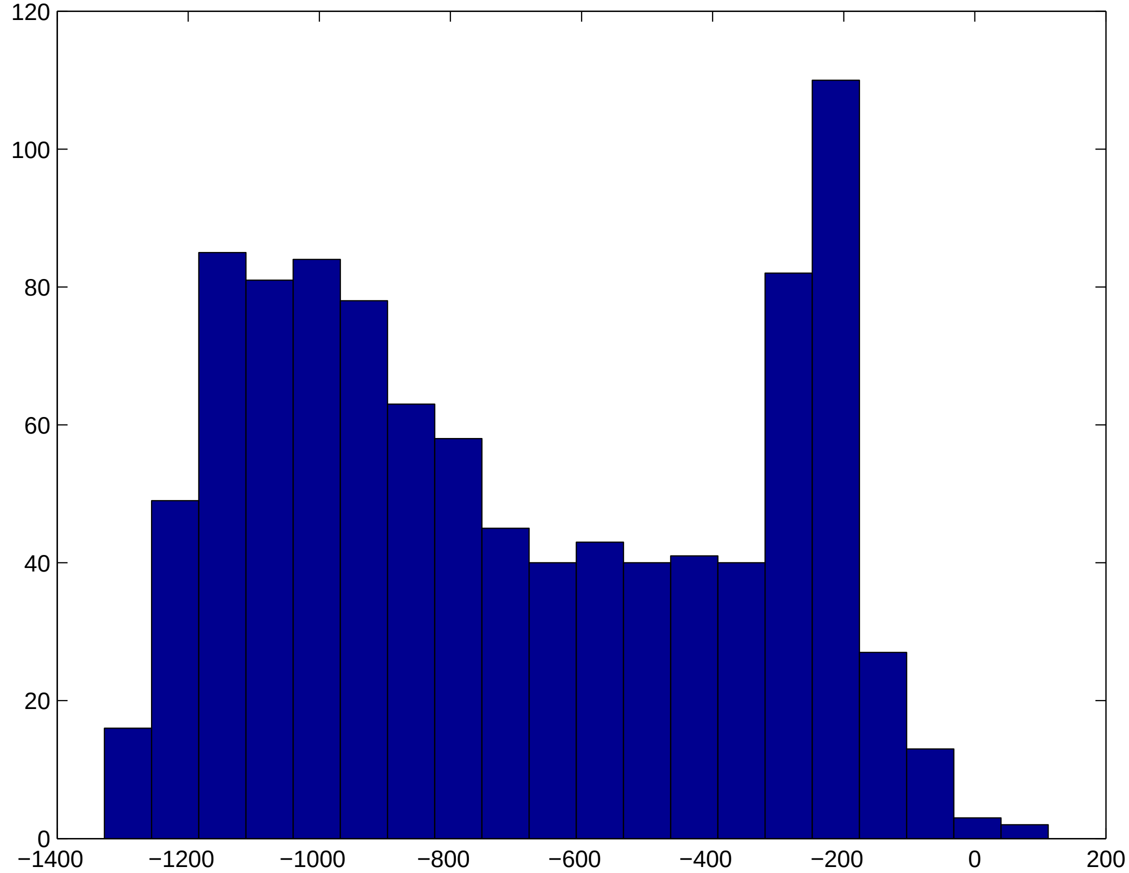
with SDP relaxation

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(WX) \\ \text{subject to} & \mathbf{diag}(X) = \mathbf{1}, X \succeq 0 \end{array}$$

and solution $X^{\text{opt}}$

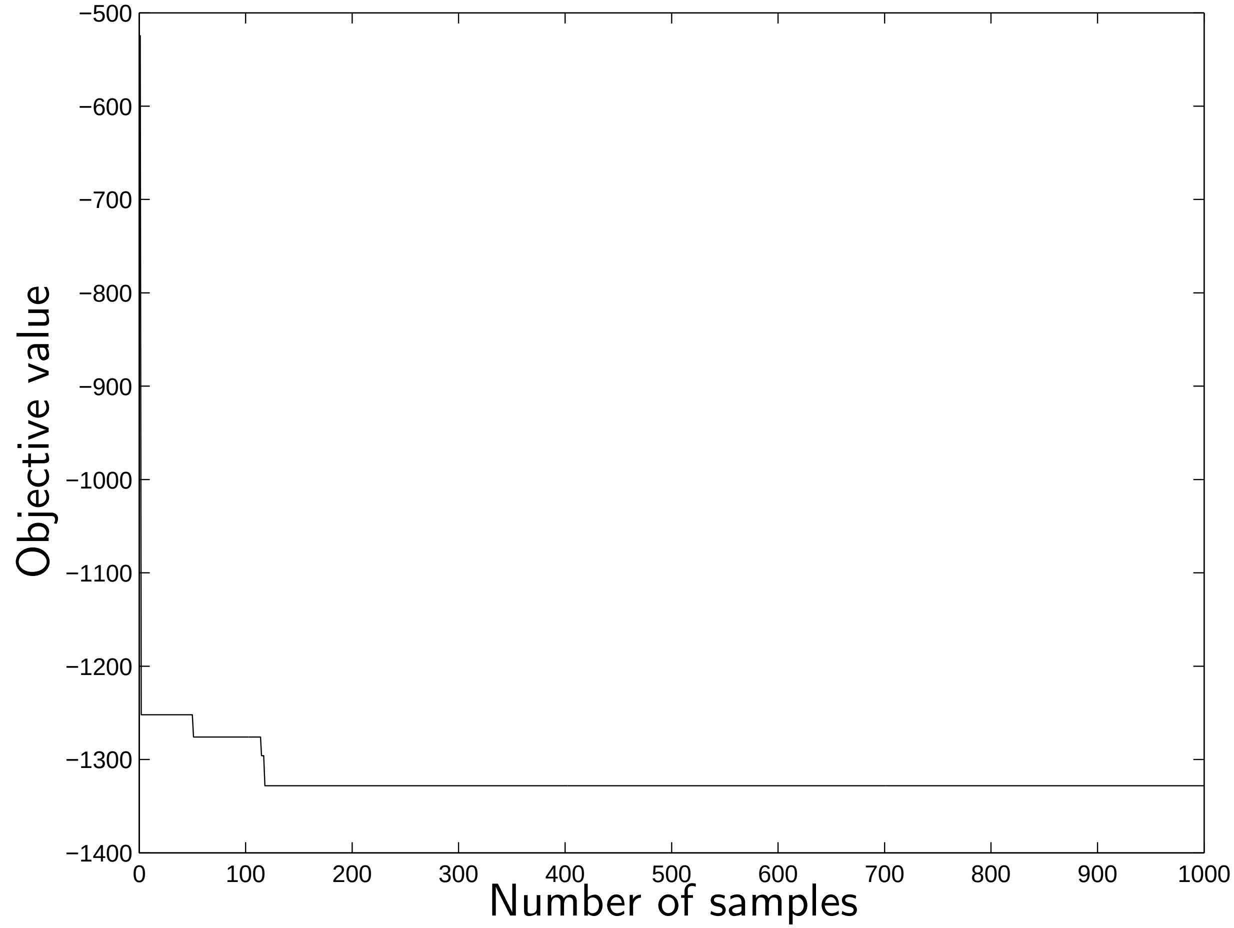
- generate samples $x^{(i)} \sim \mathcal{N}(0, X^{\text{opt}})$, $\hat{x}^{(i)} = \mathbf{sign}(x^{(i)})$
- take one with lowest cost (SDP$^{\text{opt}}$ is $-1641$)

# Histogram of partitions




heuristic on $1000$ samples: minimum value attained is $-1328$

# Objective progress in partitioning




know optimal cost is between $-1641$ and $-1328$