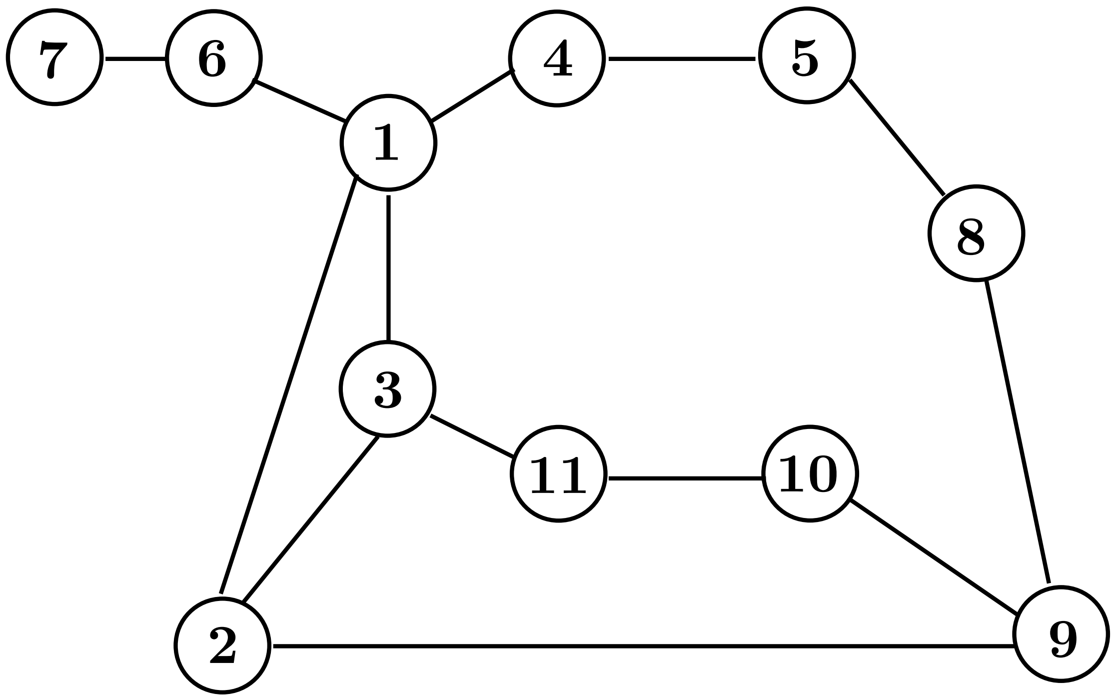
Convex-Concave Games

Example: Optimal Server Location on Network

network with 11 nodes



payoff matrix is

$$P = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 \\ 2 & 3 & 2 & 1 & 0 & 3 & 4 & 1 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 3 & 0 & 1 & 4 & 3 & 4 & 3 \\ 2 & 3 & 3 & 3 & 4 & 1 & 0 & 5 & 4 & 5 & 4 \\ 3 & 2 & 3 & 2 & 1 & 4 & 5 & 0 & 1 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 & 3 & 4 & 1 & 0 & 1 & 2 \\ 3 & 2 & 2 & 3 & 3 & 4 & 5 & 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 & 3 & 3 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}. \tag{1}$$

($P(i,j)$ is distance between nodes i and j)

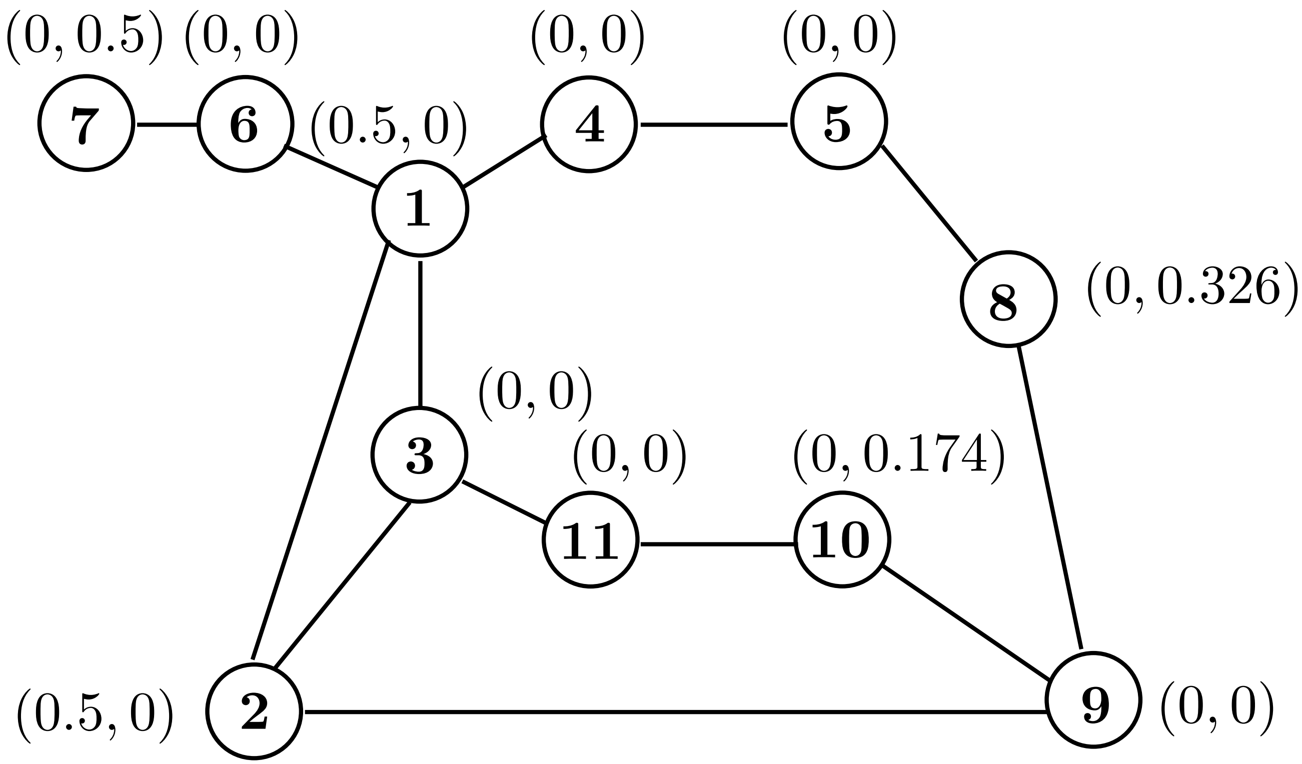
- request deterministic, but unknown: place server at center of graph (nodes 1, 2, 3 or 4); then delay no greater than 3

- request has (known) distribution v_0: solve LP

$$\begin{array}{ll} \text{minimize} & (Pv_0)^T u \\ \text{subject to} & \mathbf{1}^T u = 1. \end{array} \qquad (2)$$

  for example, if v_0 is uniform: place server at node 1 or 3 (or any combination); expected delay is 1.636

- request has unknown probability distribution: server's optimal distribution found as solution to matrix game



- pair $(u_i^\star, v_i^\star)$ next to node i specifies probability of server and request originating at node i, where $u^\star$, $v^\star$ solve the game

- solution need not be unique: another optimal server distribution is

$$\tilde{u}^\star = (0.486, 0.077, 0, 0, 0, 0.104, 0.035, 0.298, 0, 0, 0). \qquad (3)$$

Note: Not every solution to

$$\begin{array}{ll} \text{minimize} & (Pv^\star)^T u \\ \text{subject to} & \mathbf{1}^T u = 1 \end{array} \qquad (4)$$

solves the game: (bilinear) objective function is not strictly convex-concave in u and v.

An example of a convex-concave game

- m Gaussian communications channels, signal powers $p_i \geq 0$, noise (or interference) power $n_i \geq 0$
- power budget: total signal power P, total noise power N
- objective for signal powers is to maximize total capacity, for noise powers to minimize total capacity:

$$\begin{array}{lll} \text{maximize}_p & \text{minimize}_n & \sum_{i=1}^m \log(1 + \frac{\beta_i p_i}{\sigma_i + n_i}) \\ \text{subject to} & & \mathbf{1}^T p = P, \\ & & \mathbf{1}^T n = N, \\ & & p \succeq 0, n \succeq 0. \end{array} \tag{5}$$

objective is convex in n and concave in p

Specific example

specific instance with 10 channels, solved using barrier method

- $P = 20$, $N = 10$

- $\sigma = (2, 6, 5, 8, 3, 9, 5, 6, 7, 3)$

- $\beta_i = 1$, $i = 1, \ldots, m$

optimal allocation of signal powers is

$$p^\star = (2.734, 2.333, 2.733, 0.334, 2.733, 0.000, 2.733, 2.333, 1.333, 2.733),$$

worst possible noise distribution is

$$n^\star = (3.6, 0, 0.6, 0, 2.6, 0, 0.6, 0, 0, 2.6).$$

value of the game, $C^\star = 2.860$

figure shows that $p^\star$ is waterfilling solution to effective noise distribution $(\sigma + n^\star)/\beta$: