EE364b Spring 2023 Homework 8

Due Sunday 6/04 at 11:59pm via Gradescope

8.1 *(5 points) Portfolio investment with linear and fixed costs.* The goal is to optimally invest an initial amount B in a portfolio of n assests. Let x_i be the amount (in dollars) of asset i that we purchase. We assume that no short selling is allowed, *i.e.*, $x \succeq 0$. We pay a fixed cost β for each asset we invest in, *i.e.*, for each asset with $x_i > 0$. We also pay a fee that is proportional to the amount purchased, given by $\alpha^T x$, where α_i is the fee (rate) associated with asset i. The budget constraint can be expressed as

$$\mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \le B.$$

The mean return on the portfolio is given by $\mu^T x$, where μ_i is the mean return of asset i, and the variance of the portfolio is $x^T \Sigma x$, where Σ is the covariance of the price changes. Our goal is to minimize the portfolio standard deviation, subject to meeting a minimum mean return requirement. This can be expressed as the problem

$$\begin{array}{ll} \text{minimize} & (x^T \Sigma x)^{1/2} \\ \text{subject to} & \mu^T x \ge R_{\min}, \qquad x \succeq 0 \\ & \mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \le B. \end{array} \tag{1}$$

This is a convex-cardinality problem. Once the choice is made of which assets to invest in, the problem becomes convex. But there are 2^n possible subsets of assets to invest in, so exhaustive search over all of these is not practical for $n \ge 20$.

You will focus on a particular instance of this problem, with data given in the file `l1_heuristic_portfolio_data.{py,m}` at Canvas/Files/Problem Sets/hw8.

(a) *Heuristic portfolio investment.* You will use the following ℓ_1-norm based heuristic to approximately solve the portfolio investment problem. First we replace $\mathbf{card}(x)$ with $\gamma \mathbf{1}^T x$ (which is the same as $\gamma \|x\|_1$, since $x \succeq 0$), where $\gamma \ge 0$ is a parameter, to get the problem

$$\begin{array}{ll} \text{minimize} & (x^T \Sigma x)^{1/2} \\ \text{subject to} & \mu^T x \ge R_{\min}, \qquad x \succeq 0 \\ & \mathbf{1}^T x + \beta\gamma \mathbf{1}^T x + \alpha^T x \le B. \end{array} \tag{2}$$

Solve (2) for, say, 50 values of γ in $[0, 25]$. Note that when you solve the problem for $\gamma = 0$, you are just ignoring the fixed investment costs. The solutions of (2) need not be feasible for the portfolio investment problem (1), since the true budget constraint can be violated.

For each solution of (2), note the sparsity pattern of x. Fix this sparsity pattern (which makes $\mathbf{card}(x)$ constant), and solve the problem (1). This procedure is called *polishing*.

Plot the portfolio standard deviation obtained (after polishing) versus γ. What is the minimum standard deviation, $\sigma_{\min}$, that you obtain? Give the best portfolio found (*i.e.*, the assets purchased, and the amounts for each one). Also plot $\mathbf{card}(x)$, *i.e.*, the number of assets invested in, versus γ.

Hint. To determine the sparsity pattern of x after solving (2), you'll need to use a reasonable (positive) threshold to determine if $x_i = 0$, as in `find(x<1e-3)`.

(b) *A lower bound.* For some values of γ the optimal value of (2) is a lower bound on the optimal value of the original portfolio investment problem (1). Find a simple value $\tilde{\gamma}$ of γ for which this is the case. Compute the corresponding lower bound and compare it to the standard deviation found in (a).

(c) *A more sophisticated lower bound.* Let $x^\star$ denote an optimal point for the original portfolio investment problem (1). Suppose $x_i^\star \leq u_i$ for $i = 1, \ldots, n$. Explain why the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & (x^T \Sigma x)^{1/2} \\ \text{subject to} & \mu^T x \geq R_{\min}, \qquad x \succeq 0 \\ & \mathbf{1}^T x + \beta v^T x + \alpha^T x \leq B, \end{array}$$

where $v_i = 1/u_i$, gives a lower bound on the optimal value of the original portfolio investment problem (1).

A simple choice for u_i is $u_i = B$. Better (*i.e.*, smaller) values can be found as the optimal values of the problems

$$\begin{array}{ll} \text{maximize} & x_i \\ \text{subject to} & \mu^T x \geq R_{\min}, \qquad x \succeq 0 \\ & (x^T \Sigma x)^{1/2} \leq \sigma_{\min} \\ & \mathbf{1}^T x + \beta \tilde{\gamma} \mathbf{1}^T x + \alpha^T x \leq B, \end{array}$$

where $\sigma_{\min}$ is the standard deviation of the portfolio found in part (a). Justify this.

Carry out this procedure for the given problem instance, and compare the resulting lower bound to the results from parts (a) and (b).

8.2 (5 points - extra credit) *Trajectory optimization with avoidance constraints.* In this problem, you must choose N trajectories (say, of some vehicles) in $\mathbf{R}^n$, which are denoted by $p_i(t) \in \mathbf{R}^n$, $t = 1, \dots, T$, $i = 1, \dots, N$. The objective is to minimize

$$J = \sum_{i=1}^{N} \sum_{t=1}^{T-1} \|p_i(t+1) - p_i(t)\|_2^2,$$

subject to fixed starting and final positions,

$$p_i(1) = p_i^{\text{start}}, \quad p_i(T) = p_i^{\text{final}}, \quad i = 1, \dots, N.$$

The solution to the problem stated so far is simple: Each trajectory follows a straight line from the starting position to the final position, at uniform speed. Here is the wrinkle: We have *avoidance constraints*, of the form

$$\|p_i(t) - p_j(t)\|_2 \geq D, \quad i \neq j, \quad t = 2, \dots, T-1.$$

(Thus, the vehicles must maintain a given distance D from each other at all times.) These last constraints are obviously not convex.

(a) Explain how to use the convex-concave procedure (see lecture slides) to (approximately, locally) solve the problem by replacing concave functions with their affine approximations.

(b) Implement the method for the problem with data given in `traj_avoid_data.{py,m}`, which involves three vehicles moving in $\mathbf{R}^2$. Executing this file runs a movie showing the vehicle trajectories when they move in straight lines at uniform speed, with a circle around each vehicle of diameter D. You can use this code to visualize the trajectories you obtain as your algorithm runs.

You should start the convex-concave procedure from several different initial trajectories for which $p_i(t) \neq p_j(t)$, $t = 1, \dots, T$. For example, you can simply take $p_i(t) \sim \mathcal{N}(0, I)$. Plot the objective J versus iteration number for a few different initial trajectories (on the same plot). Verify that the avoidance constraints are satisfied after the first iteration.

For the best set of final trajectories you find, plot the minimum inter-vehicle distance, $\min_{i \neq j} \|p_i(t) - p_j(t)\|_2$, versus time. Plot the trajectories in $\mathbf{R}^2$. And of course, for your own amusement, view the movie.

Hint. Don't try to write elegant code that handles the case of general N. We've chosen $N = 3$ so you can just name the trajectories ($2 \times T$ matrices) `p1`, `p2`, and `p3`, and explicitly write out the three (*i.e.*, $N(N-1)/2$) avoidance constraints.