EE364a Midterm Review

Administrative info:

• midterm exam: Thursday, July 12, first hour of lecture

• closed notes, closed book (you shouldn’t need it)
Convexity

simple ideas

• sets
  – any “mixture” of points is in the set (midpoint is (usually) enough)
  – all points have “clear view” of all others
  – is a (possibly uncountably infinite) intersection of half-spaces
  – its intersection with any line is convex in one dimension

• functions
  – nonnegative curvature (upwards)
  – monotonically nondecreasing gradient (any direction)
  – always no larger than any linear interpolation (chord lies above graph)
  – is pointwise sup (max) of linear functions
Convex sets

A convex set $C$ contains its line segments between constituent points

$$x, y \in C, \quad 0 \leq \theta \leq 1 \implies \theta x + (1 - \theta) y \in C.$$ 

demonstrating convexity

- apply definition above (usually not the easiest way)
- show $C$ is obtained by simple convex sets (hyperplanes, halfspaces, balls, etc.) by convexity-preserving operations
- show $C$ is an $\alpha$-sublevel set of a convex function (for some $\alpha \in \mathbb{R}$)
Examples

consider \( f(x) = x^2 \)

\[
S = \{ x \in \mathbb{R} \mid f(x) \leq 1 \} \quad \text{Convex}
\]

\[
S = \{ x \in \mathbb{R} \mid f(x) \geq 1 \} \quad \text{Not convex}
\]

\[
S = \{ (x, f(x)) \mid x \in \mathbb{R} \} \quad \text{Not convex}
\]

\[
S = \{ (x, y) \mid y \geq f(x) \} \quad \text{Convex}
\]

\[
S = \{ x \mid f(x) \geq -1 \} \quad \text{Convex}
\]
Combinations and hulls

\[ y = \theta_1 x_1 + \cdots + \theta_k x_k \] is a

- **linear combination** of \( x_1, \ldots, x_k \)
- **affine combination** if \( \sum_i \theta_i = 1 \)
- **convex combination** if \( \sum_i \theta_i = 1, \theta_i \geq 0 \)
- **conic combination** if \( \theta_i \geq 0 \)

((linear, affine, . . .) **hull** of \( S = \{x_1, \ldots, x_k\} \) is a set of all
((linear, affine, . . .) combinations from \( S \)

- linear hull: \( \text{span}(S) \)
- affine hull: \( \text{aff}(S) \)
- convex hull: \( \text{conv}(S) \)
- conic hull: \( \text{cone}(S) \)
**example:** a few simple relations:

\[
\text{conv}(S) \subseteq \text{aff}(S) \subseteq \text{span}(S), \quad \text{conv}(S) \subseteq \text{cone}(S) \subseteq \text{span}(S).
\]

**example:** \(S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq \mathbb{R}^3\)

what is the linear hull? affine hull? convex hull? conic hull?

- **linear hull:** \(\mathbb{R}^3\).
- **affine hull:** hyperplane passing through \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).
- **convex hull:** triangle with vertices at \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).
- **conic hull:** \(\mathbb{R}_+^3\)
Important rules

- **intersection**

\[ S_\alpha \text{ is } \begin{pmatrix} \text{subspace} \\ \text{affine} \\ \text{convex} \\ \text{convex cone} \end{pmatrix} \quad \text{for } \alpha \in \mathcal{A} \implies \bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ is } \begin{pmatrix} \text{subspace} \\ \text{affine} \\ \text{convex} \\ \text{convex cone} \end{pmatrix} \]

**example:** a polyhedron is intersection of a finite number of halfspaces and hyperplanes.

- **functions that preserve convexity**
  
  **examples:** affine, perspective, and linear fractional functions.

  if \( C \) is convex, and \( f \) is an affine/perspective/linear fractional function, then \( f(C) \) is convex and \( f^{-1}(C) \) is convex.
Solution set of a quadratic inequality

let $C \subseteq \mathbb{R}^n$ be the solutions to a quadratic inequality ($A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$)

$$C = \{ x \in \mathbb{R}^n \mid x^T Ax + b^T x + c \leq 0 \},$$

*Show that $C$ is convex, provided that $A \succeq 0*

**Solution:** Intersect $C$ with an arbitrary line and show it is convex.

An arbitrary line looks like $\{ \hat{x} + tv \mid t \in \mathbb{R} \}$. Observe that

$$(\hat{x} + tv)^T A(\hat{x} + tv) + b^T (\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma$$

where: $\alpha = v^T Av$, $\beta = b^T v + 2\hat{x}^T Av$, $\gamma = c + b^T \hat{x} + \hat{x}^T A\hat{x}$.

The intersection of $C$ with the line above is thus

$$\{ \hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0 \},$$

which is convex if $\alpha \geq 0$. This is true for any $v$ if $A \succeq 0$ (why?).
Convex functions

\( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex if \( \text{dom } f \) is a convex set and

\[
f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)
\]

for all \( x, y \in \text{dom } f \), \( 0 \leq \theta \leq 1 \)
Demonstrating convexity of functions

how to determine if a function $f$ is convex?

• apply definition

• simple function known to be convex
  – affine, exponential, negative entropy, log-sum-exp, norms, . . .

• for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$

• show that $f$ is obtained by operations that preserve convexity
  – nonnegative weighted sum, composition with affine function, pointwise maximum and supremum, composition, minimization, perspective

• warning: a function $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ may be convex in one argument but not the other! (connection to Hessian)
Maximization and minimization rules

pointwise maximum and supremum versus minimization

• if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

• if $f(x, y)$ is convex in $x$ for each $y \in \mathcal{A}$, then

\[
g(x) = \sup_{y \in \mathcal{A}} f(x, y)
\]

is convex

• if $f(x, y)$ is convex in $(x, y)$ and $C$ is a convex set, then

\[
g(x) = \inf_{y \in C} f(x, y)
\]

is convex
Convexity via composition

composition of \( g : \mathbb{R}^n \to \mathbb{R} \) and \( h : \mathbb{R} \to \mathbb{R} \):

\[
  f(x) = h(g(x))
\]

\( f \) is convex if \( g \) convex, \( h \) convex, \( \tilde{h} \) nondecreasing

\( g \) concave, \( h \) convex, \( \tilde{h} \) nonincreasing

- proof (for \( n = 1 \), differentiable \( g, h \))

\[
  f''''(x) = h''''(g(x))g'(x)^2 + h'(g(x))g''(x)
\]

- note: monotonicity must hold for extended-value extension \( \tilde{h} \)

similarly, composition of \( g : \mathbb{R}^n \to \mathbb{R}^k \) and \( h : \mathbb{R}^k \to \mathbb{R} \):

\[
  f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))
\]

\( f \) is convex if \( g_i \) concave, \( h \) convex, \( \tilde{h} \) nonincreasing in each argument, etc.
Summary: operations that preserve convexity

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- convex functions and convex sets are related via the epigraph

- composition rules are extremely important (definitely will be on the quiz!)
Convex or not?

are the following functions convex, concave, or neither?

- $f(x) = \log(a^T x - b)$
  concave: concave function with affine composition

- $f(x) = \log(1 + 1/x)$
  convex: increasing convex function (log-sum-exp) composed with convex function ($-\log(x)$)

- $f^*(y) = \sup_{x \in \text{dom} f}(y^T x - f(x))$
  convex: supremum of an affine function
Another example

For $A \in S^n_+$, let $f$ be the function that returns the maximum semiaxis length of the ellipsoid defined by $S = \{x_c + Au \mid \|u\|_2 \leq 1\}$. Is this function convex?

**solution:** First, express $f(A)$ as:

$$f(A) = \lambda_{\text{max}}(A),$$

where $\lambda_{\text{max}}$ denotes the maximum eigenvalue of $A$. Equivalently, $f(A) = \max_{\|v\| = 1} v^T Av$

For a fixed $v$, consider $g_v(A) = v^T Av$. This function is linear.

Finally,

$$f(A) = \max_{\|v\| = 1} g_v(A).$$

Therefore, $f(A)$ is convex.
More examples

Is the following a convex function (with domain $x > 0$, $y + 1 > 0$)

$$f(x, y, z) = \frac{(x - z)^2}{y + 1} + \max \left( 1 - y + \frac{1}{\sqrt{x}}, e^z, 0 \right)$$

solution. the following steps show that the function is convex:

- $\frac{(x - z)^2}{y + 1}$ is composition of quadratic-over-linear function $\frac{s^2}{t}$ with affine function that maps $(x, y, z)$ to $(x - z, y + 1)$, so is convex
- $\frac{1}{\sqrt{x}}$ is a negative-power function, so convex in $x$
- $1 - y$ is affine, so $1 - y + \frac{1}{\sqrt{x}}$ is convex in $x$ and $y$
- $e^z$ is exponential, so convex in $z$
- max term is convex, since its arguments are
- sum of left and right terms is convex
ex. 3.22(b): Show that the following function is convex:

\[ f(x, u, v) = -\sqrt{uv - x^T x} \]

on \( \text{dom } f = \{ (x, u, v) \mid uv > x^T x, \ u, \ v > 0 \} \). Use the fact that \( x^T x/u \) is convex in \((x, u)\) for \( u > 0 \), and that \( -\sqrt{z_1 z_2} \) is convex on \( \mathbb{R}^2_{++} \).

solution.

- take \( f(x, u, v) = -\sqrt{u(v - x^T x/u)} \)

- \( g_1(u, v, x) = u \) and \( g_2(u, v, x) = v - x^T x/u \) are concave

- the function

\[
h(z_1, z_2) = \begin{cases} 
-\sqrt{z_1 z_2} & \text{if } z \succeq 0 \\
0 & \text{otherwise}
\end{cases}
\]

is convex and decreasing in each argument

- \( f(u, v, x) = h(g(u, v, x)) \) is convex