2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities
**Affine set**

**line** through $x_1, x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbb{R})$$

affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)
Convex set

**line segment** between $x_1$ and $x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)
**Convex combination and convex hull**

**convex combination** of $x_1, \ldots, x_k$: any point $x$ of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

**convex hull** $\text{conv } S$: set of all convex combinations of points in $S$
Convex cone

**conic (nonnegative) combination** of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

**convex cone**: set that contains all conic combinations of points in the set
Hyperplanes and halfspaces

**hyperplane:** set of the form \( \{ x \mid a^T x = b \} \) \((a \neq 0)\)

\[
\begin{align*}
    \text{hyperplane:} & \quad \{ x \mid a^T x = b \} \\
    & \quad (a \neq 0)
\end{align*}
\]

**halfspace:** set of the form \( \{ x \mid a^T x \leq b \} \) \((a \neq 0)\)

\[
\begin{align*}
    \text{halfspace:} & \quad \{ x \mid a^T x \leq b \} \\
    & \quad (a \neq 0)
\end{align*}
\]

- \(a\) is the normal vector
- hyperplanes are affine and convex; halfspaces are convex
Euclidean balls and ellipsoids

(Euclidean) ball with center $x_c$ and radius $r$:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with $P \in \mathbb{S}^n_{++}$ (i.e., $P$ symmetric positive definite)

other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$ with $A$ square and nonsingular
Norm balls and norm cones

**norm:** a function \( \| \cdot \| \) that satisfies

- \( \| x \| \geq 0; \| x \| = 0 \) if and only if \( x = 0 \)
- \( \| tx \| = |t| \| x \| \) for \( t \in \mathbb{R} \)
- \( \| x + y \| \leq \| x \| + \| y \| \)

notation: \( \| \cdot \| \) is general (unspecified) norm; \( \| \cdot \|_{\text{symb}} \) is particular norm

**norm ball** with center \( x_c \) and radius \( r \): \( \{ x \mid \| x - x_c \| \leq r \} \)

**norm cone:** \( \{ (x, t) \mid \| x \| \leq t \} \)

Euclidean norm cone is called second-order cone

norm balls and cones are convex
Polyhedra

solution set of finitely many linear inequalities and equalities

\[ Ax \preceq b, \quad Cx = d \]

\((A \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{p \times n}, \preceq \text{ is componentwise inequality})\)

polyhedron is intersection of finite number of halfspaces and hyperplanes
Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}_+^n = \{ X \in \mathbf{S}^n \mid X \succeq 0 \}$: positive semidefinite $n \times n$ matrices
  \[ X \in \mathbf{S}_+^n \iff z^T X z \geq 0 \text{ for all } z \]
- $\mathbf{S}^n_+$ is a convex cone
- $\mathbf{S}^{++}_n = \{ X \in \mathbf{S}^n \mid X \succ 0 \}$: positive definite $n \times n$ matrices

example: \[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix} \in \mathbf{S}_+^2
\]
Operations that preserve convexity

practical methods for establishing convexity of a set \( C \)

1. apply definition

\[ x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta)x_2 \in C \]

2. show that \( C \) is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions
the intersection of (any number of) convex sets is convex

example:

\[ S = \{ x \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3 \} \]

where \( p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt \)

for \( m = 2 \):
Affine function

suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$)

- the image of a convex set under $f$ is convex
  
  \[ S \subseteq \mathbb{R}^n \text{ convex} \implies f(S) = \{ f(x) \mid x \in S \} \text{ convex} \]

- the inverse image $f^{-1}(C)$ of a convex set under $f$ is convex
  
  \[ C \subseteq \mathbb{R}^m \text{ convex} \implies f^{-1}(C) = \{ x \in \mathbb{R}^n \mid f(x) \in C \} \text{ convex} \]

examples

- scaling, translation, projection
- solution set of linear matrix inequality \[ \{ x \mid x_1 A_1 + \cdots + x_m A_m \preceq B \} \]
  (with $A_i, B \in \mathbb{S}^p$)
- hyperbolic cone \[ \{ x \mid x^T P x \leq (c^T x)^2, \ c^T x \geq 0 \} \]
  (with $P \in \mathbb{S}_+^n$)
Perspective and linear-fractional function

perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$:

$$P(x, t) = x/t, \quad \text{dom } P = \{(x, t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex
example of a linear-fractional function

\[ f(x) = \frac{1}{x_1 + x_2 + 1} \]
Generalized inequalities

A convex cone $K \subseteq \mathbb{R}^n$ is a proper cone if

- $K$ is closed (contains its boundary)
- $K$ is solid (has nonempty interior)
- $K$ is pointed (contains no line)

Examples

- Nonnegative orthant $K = \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n\}$
- Positive semidefinite cone $K = S_+^n$
- Nonnegative polynomials on $[0, 1]$:

  $$K = \{x \in \mathbb{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$
generalized inequality defined by a proper cone $K$:

\[ x \preceq_K y \iff y - x \in K, \quad x \prec_K y \iff y - x \in \text{int} K \]

elements

- componentwise inequality ($K = \mathbb{R}_+^n$)

\[ x \preceq_{\mathbb{R}_+^n} y \iff x_i \leq y_i, \quad i = 1, \ldots, n \]

- matrix inequality ($K = \mathbb{S}_+^n$)

\[ X \preceq_{\mathbb{S}_+^n} Y \iff Y - X \text{ positive semidefinite} \]

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, e.g.,

\[ x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v \]
Minimum and minimal elements

$\leq_K$ is not in general a linear ordering: we can have $x \not\leq_K y$ and $y \not\leq_K x$

$x \in S$ is the minimum element of $S$ with respect to $\leq_K$ if

$$y \in S \implies x \leq_K y$$

$x \in S$ is a minimal element of $S$ with respect to $\leq_K$ if

$$y \in S, \quad y \leq_K x \implies y = x$$

example ($K = \mathbb{R}^2_+$)

$x_1$ is the minimum element of $S_1$

$x_2$ is a minimal element of $S_2$
Separating hyperplane theorem

if $C$ and $D$ are nonempty disjoint convex sets, there exist $a \neq 0$, $b$ s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$

the hyperplane $\{x \mid a^T x = b\}$ separates $C$ and $D$

strict separation requires additional assumptions (e.g., $C$ is closed, $D$ is a singleton)
Supporting hyperplane theorem

**supporting hyperplane** to set $C$ at boundary point $x_0$:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

**supporting hyperplane theorem**: if $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$
Dual cones and generalized inequalities

dual cone of a cone $K$:

$$K^* = \{ y \mid y^T x \geq 0 \text{ for all } x \in K \}$$

examples

- $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
- $K = S^n_+$: $K^* = S^n_+$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$

first three examples are self-dual cones

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$
Minimum and minimal elements via dual inequalities

**minimum element** w.r.t. $\preceq_K$

$x$ is minimum element of $S$ iff for all $\lambda \succ_K 0$, $x$ is the unique minimizer of $\lambda^T z$ over $S$

**minimal element** w.r.t. $\preceq_K$

- if $x$ minimizes $\lambda^T z$ over $S$ for some $\lambda \succ_K 0$, then $x$ is minimal
  - if $x$ is a minimal element of a convex set $S$, then there exists a nonzero $\lambda \succeq_K 0$ such that $x$ minimizes $\lambda^T z$ over $S$
optimal production frontier

- different production methods use different amounts of resources $x \in \mathbb{R}^n$
- production set $P$: resource vectors $x$ for all possible production methods
- efficient (Pareto optimal) methods correspond to resource vectors $x$ that are minimal w.r.t. $\mathbb{R}^n_+$

example ($n = 2$)

$x_1, x_2, x_3$ are efficient; $x_4, x_5$ are not