

Disciplined Convex Programming and CVX

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Convex optimization solvers

- **LP solvers**
 - lots available (GLPK, Excel, Matlab's `linprog`, . . .)
- **cone solvers**
 - typically handle (combinations of) LP, SOCP, SDP cones
 - several available (SDPT3, SeDuMi, CSDP, . . .)
- **general convex solvers**
 - some available (CVXOPT, MOSEK, . . .)
- plus lots of special purpose or application specific solvers
- you will write your own by the end of the course!

Transforming problems to standard form

- there are lots of tricks for transforming a problem into an equivalent one that has a standard form (*e.g.*, LP, SDP)
 - introducing slack variables
 - introducing new variables that upper bound expressions
- these tricks greatly extend the applicability of standard solvers
- writing code to carry out this transformation is often painful
- **modeling systems** can partly automate this step

Modeling systems

a typical modeling system

- automates most of the transformation to standard form; supports
 - declaring optimization variables
 - describing the objective function
 - describing the constraints
 - choosing (and configuring) the solver
- when given a problem instance, calls the solver
- interprets and returns the solver's status (optimal, infeasible, . . .)
- (when solved) transforms the solution back to original form

Disciplined convex programming

- describe objective and constraints using expressions formed from
 - a set of basic atoms (convex, concave functions)
 - a restricted set of operations or rules (that preserve convexity)
- modeling system keeps track of affine, convex, concave expressions
- rules ensure that
 - expressions recognized as convex (concave) are convex (concave)
 - but, some convex (concave) expressions are not recognized as convex (concave)
- problems described using DCP are convex by construction

CVX

- by Stephen Boyd and Michael Grant
- uses DCP
- runs in Matlab, between the `cvx_begin` and `cvx_end` commands
- relies on SDPT3 or SeDuMi (LP/SOCP/SDP) solvers
- refer to user guide, online help for more info
- the CVX example library has more than a hundred examples

Example: Constrained norm minimization

```
A = randn(5, 3);  
b = randn(5, 1);  
cvx_begin  
    variable x(3);  
    minimize(norm(A*x - b, 1))  
    subject to  
        -0.5 <= x;  
        x <= 0.3;  
cvx_end
```

- between `cvx_begin` and `cvx_end`, `x` is a CVX variable
- statement `subject to` does nothing, but can be added for readability
- inequalities are interpreted elementwise

What CVX does

after `cvx_end`, CVX

- transforms problem into an LP
- calls solver SDPT3
- overwrites (object) `x` with (numeric) optimal value
- assigns problem optimal value to `cvx_optval`
- assigns problem status (which here is `Solved`) to `cvx_status`

(had problem been infeasible, `cvx_status` would be `Infeasible` and `x` would be `NaN`)

Transforming to LP examples

- minimize $\|Ax - b\|_\infty$

Transforming to LP examples

- minimize $\|Ax - b\|_\infty$
- equivalent to the LP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & Ax - b \geq -t\mathbf{1} \end{array}$$

Transforming to LP examples contd.

- minimize $\|Ax - b\|_1$

Transforming to LP examples contd.

- minimize $\|Ax - b\|_1$
- equivalent to the LP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \leq s \\ & Ax - b \geq -s \end{array}$$

Variables and affine expressions

- declare variables with variable name[(dims)] [attributes]
 - variable `x(3)`;
 - variable `C(4,3)`;
 - variable `S(3,3)` symmetric;
 - variable `D(3,3)` diagonal;
 - variables `y z`;
- can use standard matlab syntax with cvx variables
 - `3*x + 4`
 - `A*x - y`
 - `x(2:3)`
 - `X(3,5)`
 - `sum(x)`
 - **for** loops

Objectives and constraints

- **objective** can be
 - minimize(convex expression)
 - maximize(concave expression)
 - omitted (feasibility problem)
- **constraints** can be
 - convex expression \leq concave expression
 - concave expression \geq convex expression
 - affine expression $=$ affine expression
 - omitted (unconstrained problem)

Some functions

function	meaning	attributes
<code>norm(x, p)</code>	$\ x\ _p$	cvx
<code>square(x)</code>	x^2	cvx
<code>square_pos(x)</code>	$(x_+)^2$	cvx, nondecr
<code>pos(x)</code>	x_+	cvx, nondecr
<code>sum_largest(x, k)</code>	$x_{[1]} + \dots + x_{[k]}$	cvx, nondecr
<code>sqrt(x)</code>	$\sqrt{x} \quad (x \geq 0)$	ccv, nondecr
<code>inv_pos(x)</code>	$1/x \quad (x > 0)$	cvx, nonincr
<code>max(x)</code>	$\max\{x_1, \dots, x_n\}$	cvx, nondecr
<code>quad_over_lin(x, y)</code>	$x^2/y \quad (y > 0)$	cvx, nonincr in y
<code>lambda_max(X)</code>	$\lambda_{\max}(X) \quad (X = X^T)$	cvx
<code>huber(x)</code>	$\begin{cases} x^2, & x \leq 1 \\ 2 x - 1, & x > 1 \end{cases}$	cvx

Composition rules

- can combine atoms using valid composition rules, *e.g.*:
 - a convex function of an affine function is convex
 - the negative of a convex function is concave
 - a convex, nondecreasing function of a convex function is convex
 - a concave, nondecreasing function of a concave function is concave
- for convex h , $h(g_1, \dots, g_k)$ is recognized as convex if, for each i ,
 - g_i is affine, or
 - g_i is convex and h is nondecreasing in its i th arg, or
 - g_i is concave and h is nonincreasing in its i th arg
- for concave h , $h(g_1, \dots, g_k)$ is recognized as concave if, for each i ,
 - g_i is affine, or
 - g_i is convex and h is nonincreasing in i th arg, or
 - g_i is concave and h is nondecreasing in i th arg

Valid (recognized) examples

u, v, x, y are scalar variables; X is a symmetric 3×3 variable

- convex:

- `norm(A*x - y) + 0.1*norm(x, 1)`
- `quad_over_lin(u - v, 1 - square(v))`
- `lambda_max(2*X - 4*eye(3))`
- `norm(2*X - 3, 'fro')`

- concave:

- `min(1 + 2*u, 1 - max(2, v))`
- `sqrt(v) - 4.55*inv_pos(u - v)`

Rejected examples

u, v, x, y are scalar variables

- neither convex nor concave:
 - $\text{square}(x) - \text{square}(y)$
 - $\text{norm}(A*x - y) - 0.1*\text{norm}(x, 1)$
- rejected due to limited DCP ruleset:
 - $\text{sqrt}(\text{sum}(\text{square}(x)))$ (is convex; could use $\text{norm}(x)$)
 - $\text{square}(1 + x^2)$ (is convex; could use $\text{square_pos}(1 + x^2)$, or $1 + 2*\text{pow_pos}(x, 2) + \text{pow_pos}(x, 4)$)

More Examples

- $\text{norm}([x + 2*y, x - y]) == 0$
- $1/x + 1/y \leq 1; x \geq 0; y \geq 0$
- $\text{norm}([\max(x, 1), \max(y, 2)]) \leq 3*x + y$

More Examples

- $\text{norm}([x + 2*y, x - y]) == 0 \Rightarrow$

$$x==0; y==0$$

- $1/x + 1/y \leq 1; x \geq 0; y \geq 0 \Rightarrow$

$$\text{inv_pos}(x) + \text{inv_pos}(y) \leq 1$$

- $\text{norm}([\max(x, 1), \max(y, 2)]) \leq 3*x + y \Rightarrow$

$$\text{norm}([u, v]) \leq 3*x + y$$

$$\max(x, 1) \leq u$$

$$\max(y, 2) \leq v$$

Sets

- some constraints are more naturally expressed with convex sets
- sets in CVX work by creating unnamed variables constrained to the set
- examples:
 - `semidefinite(n)`
 - `nonnegative(n)`
 - `simplex(n)`
 - `lorentz(n)`
- `semidefinite(n)`, say, returns an unnamed (symmetric matrix) variable that is constrained to be positive semidefinite

Using the semidefinite cone

variables: X (symmetric matrix), z (vector), t (scalar)

constants: A and B (matrices)

- $X == \text{semidefinite}(n)$
 - means $X \in \mathbf{S}_+^n$ (or $X \succeq 0$)
- $A*X*A' - X == B*\text{semidefinite}(n)*B'$
 - means $\exists Z \succeq 0$ so that $AXA^T - X = BZB^T$
- $[X \ z; \ z' \ t] == \text{semidefinite}(n+1)$
 - means $\begin{bmatrix} X & z \\ z^T & t \end{bmatrix} \succeq 0$

More involved example

```
A = randn(5);
A = A'*A;
cvx_begin
    variable X(5, 5) symmetric;
    variable y;
    minimize(norm(X) - 10*sqrt(y))
    subject to
        X - A == semidefinite(5);
        X(2,5) == 2*y;
        X(3,1) >= 0.8;
        y <= 4;
cvx_end
```

Defining new functions

- can make a new function using existing atoms
- **example:** the convex deadzone function

$$f(x) = \max\{|x| - 1, 0\} = \begin{cases} 0, & |x| \leq 1 \\ x - 1, & x > 1 \\ 1 - x, & x < -1 \end{cases}$$

- create a file `convex_deadzone.m` with the code

```
function y = convex_deadzone(x)
y = max(abs(x) - 1, 0)
```

- `convex_deadzone` makes sense both within and outside of CVX

Defining functions via incompletely specified problems

- suppose f_0, \dots, f_m are convex in (x, z)
- let $\phi(x)$ be optimal value of convex problem, with variable z and parameter x

$$\begin{array}{ll} \text{minimize} & f_0(x, z) \\ \text{subject to} & f_i(x, z) \leq 0, \quad i = 1, \dots, m \\ & A_1x + A_2z = b \end{array}$$

- ϕ is a convex function
- problem above sometimes called *incompletely specified* since x isn't (yet) given
- an incompletely specified concave maximization problem defines a concave function

CVX functions via incompletely specified problems

```
implement in cvx with
function cvx_optval = phi(x)
cvx_begin
    variable z;
    minimize(f0(x, z))
    subject to
        f1(x, z) <= 0; ...
        A1*x + A2*z == b;
cvx_end
```

- function `phi` will work for numeric `x` (by solving the problem)
- function `phi` can also be used inside a CVX specification, wherever a convex function can be used

A more complex example

- $f(x) = x + x^{1.5} + x^{2.5}$, with $\text{dom } f = \mathbf{R}_+$, is a convex, monotone increasing function
- its inverse $g = f^{-1}$ is concave, monotone increasing, with $\text{dom } g = \mathbf{R}_+$
- there is no closed form expression for g
- $g(y)$ is optimal value of problem

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & t_+ + t_+^{1.5} + t_+^{2.5} \leq y \end{array}$$

(for $y < 0$, this problem is infeasible, so optimal value is $-\infty$)

- implement as

```
function cvx_optval = g(y)
```

```
cvx_begin
```

```
    variable t;
```

```
    maximize(t)
```

```
    subject to
```

```
        pos(t) + pow_pos(t, 1.5) + pow_pos(t, 2.5) <= y;
```

```
cvx_end
```

- use it as an ordinary function, as in g(14.3), or within CVX as a concave function:

```
cvx_begin
```

```
    variables x y;
```

```
    minimize(quad_over_lin(x, y) + 4*x + 5*y)
```

```
    subject to
```

```
        g(x) + 2*g(y) >= 2;
```

```
cvx_end
```

CVX hints/warnings

- watch out for `=` (assignment) versus `==` (equality constraint)
- `X >= 0`, with matrix `X`, is an elementwise inequality
- `X >= semidefinite(n)` means: `X` is elementwise larger than some positive semidefinite matrix (which is likely not what you want)
- writing `subject to` is unnecessary (but can look nicer)
- use brackets around objective functions:
 - **yes**: `minimize (c'*x)`
 - **no**: `minimize c'*x`
- double inequalities like `0 <= x <= 1` don't work; use `0 <= x; x <= 1` instead
- `log`, `exp`, entropy-type functions implemented using experimental successive approximation method, which can be slow, unreliable
- update CVX regularly.