EE364a Review Session 1: Convex Sets

April 7, 2009

In this review session we'll cover some examples on convex sets. In particular, we'll go over the the concept of affine/convex/conic hulls, functions that preserve convexity, and convex cones.

Announcements:

- TA office hours: Tuesdays 6:15–8:15pm, in Packard 109, Wednesdays and Thursdays 4–8pm, in Packard 277.
- Homeworks due *fridays* by 5pm.

Affine/Convex/Conic combinations

Let $x_1, \ldots, x_k$ be a set of points in $\mathbf{R}^n$. A weighted combination of these points, $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$, where $\theta_1, \ldots, \theta_k \in \mathbf{R}$, is

- a linear combination of $x_1, \ldots, x_k$.
- an affine combination of $x_1, \ldots, x_k$ if $\sum_i \theta_i = 1$.
- a convex combination of $x_1, \ldots, x_k$ if $\sum_i \theta_i = 1$, and $\theta_i \geq 0$.
- a conic combination of $x_1, \ldots, x_k$ if $\theta_i \geq 0$.

The *linear hull* of a set S is the set of all linear combinations of points in S (*i.e.*, span(S)). Similarly, the *affine/convex/conic hull* of S is the set of all affine/convex/conic combinations of points in S. For example, the convex hull of S is

$$\mathbf{conv}(S) = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in S,\ \theta_i \geq 0,\ i = 1, \ldots, k,\ \theta_1 + \cdots + \theta_k = 1\ \}.$$

An equivalent definition of a linear/affine/convex/conic hull is the smallest subspace/affine set/convex set/convex cone that contains the set.

As a very simple example, let's take two points $x_1 \in \mathbf{R}^2$ and $x_2 \in \mathbf{R}^2$. The linear hull of x_1 and x_2 is the subspace spanned by the two points, span$\{x_1, x_2\}$. (Thus, if the two vectors are linearly independent, the linear hull is simply $\mathbf{R}^2$.) The affine hull is the line that passes through x_1 and x_2, the convex hull is the line segment between x_1 and x_2, and the conic hull is pie slice between x_1 and x_2.

Example: Let S be a convex set. What is the convex hull of S? *Answer:* The convex hull of S is S, since S is the smallest convex set that contains S. Thus, a convex set contains all convex combinations of its elements.

Example: Let S be a cone. (This means that if $x \in S$, then $\theta x \in S$, where $\theta \geq 0$.) Is it true that the conic hull of S is S? *Answer:* No, this is false in general. A cone need not be convex, but the conic hull of set of points must be a convex cone.

Example: Let $S = \{e_1, e_2, e_3\} \subseteq \mathbf{R}^3$. What is the linear hull of S? What about the affine hull, convex hull, and conic hull? *Answer:* The linear hull, or in other words, the span of the points in S, is $\mathbf{R}^3$. The affine hull is the hyperplane passing through e_1, e_2, and e_3. The convex hull is the triangle with vertices at e_1, e_2, and e_3, and conic hull is the nonnegative orthant, $\mathbf{R}^3_+$.

One of the reasons we spend so much time studying convex sets is that many constraints that arise in real-world applications can be represented by convex sets. Take for example, a mechanical system driven by an actuator. The actuator can deliver a force f, which must be nonnegative (the actuator cannot move backwards) and cannot exceed a maximum force $f_{\max}$. In this case, the set of forces the actuator can deliver is $\{f \mid 0 \leq f \leq f_{\max}\}$, which is a convex set (in fact, it's an interval).

Here's another example. Suppose we have a measurement system

$$y = 0.1\mathbf{floor}(10Ax),$$

where $x \in \mathbf{R}^n$ is the input, $y \in \mathbf{R}^m$ is the measurement, and $A \in \mathbf{R}^{m\times n}$. Here, the system quantizes the measurements to steps of size 0.1. Given a measurement y, we'd like to find the set of inputs that are consistent with the measurements. This is the set

$$\mathcal{X} = \{x \mid 0 \leq a_i^T x - y_i \leq 0.1,\ i = 1, \ldots, n\},$$

i.e., all the points that lie within the quantization step size. It is not difficult to show that $\mathcal{X}$ is a polyhedron, which is a convex set (make sure you can do this yourself!).

An empirical way in which you can explore whether a set is convex is through extensive simulation. For example, if you suspect that your set is not convex, you can randomly pick points in the set, and check whether their midpoints are members of the set. Of course, this does not always work, but it's often an effective way to find counterexamples.

Operations that preserve convexity

It is often possible to determine whether a set is convex by invoking the definition, but sometimes this can get cumbersome. Another way to establish convexity of a set is by expressing it as the result of a sequence of convexity-preserving operations on a known convex set. (We will also learn a third method, when we study convex functions.)

Here's a summary of important rules that you should remember.

- The intersection of *any* number of convex sets is convex.
- If C is convex and f is an affine function, then $f(C)$ and $f^{-1}(C)$ are both convex.
- If C is convex and f is a perspective function, then $f(C)$ and $f^{-1}(C)$ are both convex.
- If C is convex and f is a linear fractional function, then $f(C)$ and $f^{-1}(C)$ are both convex.

As a simple example, let's take a polyhedron, $P = \{x \mid Ax \preceq b\} \subseteq \mathbf{R}^n$ (here the inequality is elementwise). We can think of this either as an intersection of halfspaces, or as the inverse image of the convex cone $\mathbf{R}^n_+$ under the affine function $f(x) = b - Ax$. To see this, notice we can write

$$P = \{x \mid Ax \preceq b\} = \{x \mid b - Ax \succeq 0\} = \{x \mid b - Ax \in \mathbf{R}^n_+\}.$$

Now let's do a slightly more complicated example from your homework.

Example: Show that the following set is convex: $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

Solution. $x + S_2 \subseteq S_1$ if $x + y \in S_1$ for all $y \in S_2$. Therefore

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y).$$

Since $S_1 - y$ is convex for any y, the set $\{x \mid x + S_2 \subseteq S_1\}$ must be convex, since we've just expressed it as an intersection of convex sets.

Example: Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality

$$C = \{x \in \mathbf{R}^n \mid x^T Ax + b^T x + c \leq 0\},$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$ and $c \in \mathbf{R}$. Show that C is convex if $A \succeq 0$.

Solution. For this problem, we'll use yet another trick. We'll show that C is convex by showing that the intersection of C with an arbitrary line $L = \{\hat{x} + tv \mid t \in \mathbf{R}\}$ is convex. Substituting $\hat{x} + tv$ for x, we get

$$(\hat{x} + tv)^T A(\hat{x} + tv) + b^T(\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma$$

where,

$$\alpha = v^T Av, \quad \beta = b^T v + 2\hat{x}^T Av, \quad \gamma = c + b^T \hat{x} + \hat{x}^T A\hat{x}.$$

The intersection of C with L is the set

$$\{\hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0\}$$

which is convex if $\alpha \geq 0$ (why?). This is true for any v if $A \succeq 0$.

Convex cones are an especially important class of convex sets that arise frequently in practice. Many of the convex sets you'll encounter in this class can be expressed as the image, or inverse image of a convex cone under an affine transformation. You've already seen the example of a set of linear inequalities, which can be expressed as the inverse image of the nonnegative orthant under an affine mapping. Another example is the set

$$\{x \in \mathbf{R}^n \mid F_0 + x_1 F_1 + \cdots + x_n F_n \succeq 0\},$$

where $F_0, \ldots, F_n \in \mathbf{S}^n$, and $\succeq$ is with respect to the positive semidefinite cone. This set is convex: it is the inverse image of the positive semidefinite cone $\mathbf{S}^n_+$, under an affine transformation. We should also mention that inequalities of the form

$$F_0 + x_1 F_1 + \cdots + x_n F_n \succeq 0,$$

are called linear matrix inequalities (LMIs), and arise in many applications. We'll see some of these applications in a few lectures time.