EE364a Review Session 5

overview:

- review of duality
- strong duality
- complementary slackness
- solving primal via dual
- theorems of alternatives

Duality

- primal problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- Lagrangian: $L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$
- dual function: $g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$
- for $\lambda \succeq 0$, $g(\lambda, \nu) \leq p^\star$
- dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

Strong duality

- weak duality: $d^\star \leq p^\star$ (always holds)
- strong duality: $d^\star = p^\star$ (usually holds for convex problems)
- slater's condition: strong duality holds for a convex problem if it is strictly feasible, $i.e.$

$$\exists x \in \mathbf{int}\, \mathcal{D} : \qquad f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

- affine inequality constraints do not need to be strictly feasible
- convex problem for which strong duality fails

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq 0 \end{array}$$

with domain $\mathcal{D} = \{(x, y) \mid y > 0\}$. here, $p^\star = 1$, $d^\star = 0$.

Complementary slackness

let $x^\star$ and $(\lambda^\star, \nu^\star)$ be primal and dual optimal points, and assume strong duality holds (*i.e.*, $p^\star = d^\star$). then,

$$\begin{array}{rcl} f_0(x^\star) & = & g(\lambda^\star, \nu^\star) \\ & = & \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \nu_i^\star h_i(x) \right) \\ & \leq & f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \nu_i^\star h_i(x^\star) \\ & \leq & f_0(x^\star). \end{array}$$

- $x^\star$ minimizes $L(x, \lambda^\star, \nu^\star)$ over x
- $\lambda_i^\star f_i(x^\star) = 0$ for $i = 1, \ldots, m$ (complementary slackness):

$$\lambda^\star > 0 \Longrightarrow f_i(x^\star) = 0, \qquad f_i(x^\star) < 0 \Longrightarrow \lambda_i^\star = 0$$

Example: minimizing a linear function over a rectangle

let's take a look at a familiar problem,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & 0 \preceq x \preceq \mathbf{1} \end{array}$$

- what is the solution?
  answer. if $c_i > 0$, then $x_i^\star = 0$, if $c_i < 0$ then $x_i^\star = 1$, and if $c_i = 0$, then $0 \leq x_i^\star \leq 1$
- Lagrangian is

$$L(x, \lambda, \mu) = c^T x - \lambda^T x + \mu^T (x - \mathbf{1})$$

- dual function is

$$g(\lambda, \mu) = \begin{cases} -\mathbf{1}^T \mu & c = \lambda - \mu \\ -\infty & \text{otherwise} \end{cases}$$

- suppose $c_i > 0$, what can we say about $\lambda_i^\star$ $x_i^\star$ and $\mu_i^\star$?
  answer. $\lambda_i^\star > 0$, so by complementary slackness $x_i^\star = 0$, and therefore $\mu_i^\star = 0$.

- suppose $c_i < 0$, what can we say about $\lambda_i^\star$ $x_i^\star$ and $\mu_i^\star$?
  answer. $\mu_i^\star > 0$, so by complementary slackness $x_i^\star = 1$, and therefore $\lambda_i^\star = 0$.

- what about when $c_i = 0$?
  answer. then $\lambda_i^\star - \mu_i^\star = 0$. but we cannot have both $\lambda_i^\star > 0$ and $\mu_i^\star > 0$, since that would imply both $x_i = 0$ and $x_i = 1$. as a result we must have $\lambda_i^\star = 0$ and $\mu_i^\star = 0$.

- can we write down $\lambda^\star$ and $\mu^\star$?
  answer. yes, $\lambda^\star$ is the positive part of c, and $\mu^\star$ is the negative part of c. *i.e.*, $\lambda^\star = \max(c, 0)$, $\mu^\star = \min(c, 0)$.

Solving the primal via the dual

suppose strong duality holds, $i.e.$, we have $p^\star = d^\star$, and both $p^\star$ and $d^\star$ are achieved

- how can we find a primal optimal point $x^\star$, from a dual optimal point $(\lambda^\star, \nu^\star)$?

consider the Lagrangian, evaluated at $(\lambda^\star, \nu^\star)$,

$$L(x, \lambda^\star, \nu^\star) = f_0(x) + \sum_{i=1}^{m} \lambda_i^\star f_i(x) + \sum_{i=1}^{p} \nu_i^\star h_i(x)$$

suppose the minimizer of $L(x, \lambda^\star, \nu^\star)$, $\tilde{x}$, is unique. since $x^\star$ minimizes $L(x, \lambda^\star, \nu^\star)$, then we must have $\tilde{x} = x^\star$

Example: least-norm solution over a polyhedron

consider the problem,

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax \preceq b, \end{array}$$

with $x \in \mathbf{R}^{1000}$, $A \in \mathbf{R}^{10\times 1000}$. we will assume that A is full rank

- the Lagrangian is

$$L(x, \lambda) = x^T x + \lambda^T (Ax - b),$$

  with minimizer $x = -(1/2)A^T\lambda$

- dual problem is equivalent to

$$\begin{array}{ll} \text{maximize} & -(1/4)\lambda^T A A^T \lambda - b^T\lambda \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- does strong duality hold?
  answer. yes, because we can solve $Ax = b$

- suppose $\lambda^\star$ is optimal for the dual problem, can we find a primal optimal point? and if so, is it unique?
  answer. yes, $\tilde{x} = -(1/2)A^T\lambda^\star$ minimizes the Lagrangian, but $\tilde{x}$ is unique, so $\tilde{x} = x^\star$

- for the previous example, we can uniquely construct a primal optimal point $x^\star$, given a dual optimal point $\lambda^\star$

- how many variables does the primal problem have? **answer.** 1000

- how many variables does the dual problem have? **answer.** 10

- this is one advantage of solving the dual problem and then constructing the solution of the primal problem from the solution of the dual. we will see later however, that if we fully exploit the structure of the primal problem, the time taken to solve these two problems are approximately the same

Theorems of alternatives

- we want determine the feasibility of the following system of (not necessarily convex) inequalities and equalities

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad h_i(x) = 0, \quad i = 1, \ldots, p$$

- we can write this problem as

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

  where $p^\star = 0$ if feasible and $p^\star = \infty$ if infeasible

- the dual function is

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

- $g(\lambda, \nu)$ is homogeneous in λ and ν, thus,

$$d^\star = \begin{cases} \infty & \lambda \succeq 0,\ g(\lambda, \nu) > 0 \text{ is feasible} \\ 0 & \lambda \succeq 0,\ g(\lambda, \nu) > 0 \text{ is infeasible.} \end{cases}$$

- if primal inequalities are feasible then $p^\star = 0$, thus $d^\star = 0$, so the dual inequalities

$$\lambda \succeq 0, \qquad g(\lambda, \nu) > 0$$

are infeasible

- if the dual inequalities are feasible then $d^\star = \infty$, so $p^\star = \infty$, $i.e.$, the primal system is infeasible

- both systems can be infeasible (then $d^\star = 0$ and $p^\star = \infty$)

Weak and strong alternatives

- the two systems of inequalities

$$f_i(x) \le 0, \quad i = 1, \ldots, m, \qquad h_i(x) = 0, \quad i = 1, \ldots, p$$

  and

$$\lambda \succeq 0, \qquad g(\lambda, \nu) > 0$$

  are called *weak alternatives*, since at most one of the two is feasible

- two systems are called *strong alternatives* if exactly one of the two alternatives holds

- for theorems of alternatives, whether or not the system of inequalities is strict makes a difference

Farkas' lemma

the system of inequalities

$$Ax \preceq 0, \quad c^T x < 0,$$

where $A \in \mathbf{R}^{m \times n}$, $c \in \mathbf{R}^n$, and the system of inequalities

$$A^T y + c = 0, \quad y \succeq 0,$$

are strong alternatives

- how can we show this?

solution. consider the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq 0, \end{array}$$

and its dual

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & A^T y + c = 0 \\ & y \succeq 0 \end{array}$$

- primal LP is homogeneous, so $p^\star = 0$ if primal inequality system is infeasible, and $p^\star = -\infty$ if primal inequality system is feasible

- dual LP has $d^\star = 0$ is dual inequality system is feasible, and $d^\star = -\infty$ if dual inequality system is infeasible

- since $p^\star = d^\star$, the two system are strong alternatives

Application: arbitrage

- n assets with prices $p_1, p_2, \ldots, p_n$
- at the end of the investment period the value of the assets is $v_1, v_2, \ldots, v_n$
- $x_1, x_2, \ldots, x_n$ is the initial investment in each asset ($x_j < 0$ means we owe $-x_j$ amount of asset j)
- cost of investment is $p^T x$, final value of the investment is $v^T x$
- v is uncertain, there are m possible scenarios, $v^{(1)}, \ldots, v^{(m)}$. in scenario i, the final value of the investment is $v^{(i)T} x$
- an *arbitrage* exists if there is an investment x with $p^T x < 0$, and $v^{(i)T} x \geq 0$, for $i = 1, \ldots, m$

- in finance, it is often assumed that no arbitrage exists. which means that the system of inequalities

$$Vx \succeq 0, \quad p^T x < 0$$

is infeasible (V is a matrix with rows $v^{(1)T}, \ldots, v^{(m)T}$)

- by Farkas' lemma, the above system is infeasible if and only if there exists y such that

$$V^T y = p, \quad y \succeq 0$$

- suppose that V is known, and all the prices except the last price p_n is known. we wish to find the set of prices p_n that are consistent with the no-arbitrage assumption. what kind of set is this?
solution. clearly, we must have $p \succeq 0$, which is consistent with intuition. the set must therefore be an interval (it is the projection of a polyhedron). we can find this interval by solving a pair of LPs. to find

the minimum p_n we solve,

$$\begin{array}{ll} \text{minimize} & p_n \\ \text{subject to} & V^T y = p \\ & y \succeq 0 \end{array}$$

to find the maximum p_n, we solve the same LP with maximization.