EE364a Homework 4 solutions

4.15 *Relaxation of Boolean LP.* In a *Boolean linear program*, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i=1,\ldots,n. \end{array} \tag{1}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points).

In a general method called *relaxation*, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leq x_i \leq 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i=1,\ldots,n. \end{array} \tag{2}$$

We refer to this problem as the *LP relaxation* of the Boolean LP (1). The LP relaxation is far easier to solve than the original Boolean LP.

(a) Show that the optimal value of the LP relaxation (2) is a lower bound on the optimal value of the Boolean LP (1). What can you say about the Boolean LP if the LP relaxation is infeasible?

(b) It sometimes happens that the LP relaxation has a solution with $x_i \in \{0,1\}$. What can you say in this case?

Solution.

(a) The feasible set of the relaxation includes the feasible set of the Boolean LP. It follows that the Boolean LP is infeasible if the relaxation is infeasible, and that the optimal value of the relaxation is less than or equal to the optimal value of the Boolean LP.

(b) The optimal solution of the relaxation is also optimal for the Boolean LP.

4.16 *Minimum fuel optimal control.* We consider a linear dynamical system with state $x(t) \in \mathbf{R}^n$, $t=0,\ldots,N$, and actuator or input signal $u(t) \in \mathbf{R}$, for $t=0,\ldots,N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0,\ldots,N-1,$$

where $A \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$ are given. We assume that the initial state is zero, *i.e.*, $x(0)=0$.

The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\rm des}$, where N is the (given) time horizon, and $x_{\rm des} \in \mathbf{R}^n$ is the (given) desired final or target state. The function $f : \mathbf{R} \to \mathbf{R}$ is the *fuel use map* for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

Formulate the minimum fuel optimal control problem as an LP.

Solution. The minimum fuel optimal control problem is equivalent to the LP

$$\begin{array}{ll} \mbox{minimize} & \mathbf{1}^T t \\ \mbox{subject to} & Hu = x_{\rm des} \\ & -y \preceq u \preceq y \\ & t \succeq y \\ & t \succeq 2y - \mathbf{1}, \end{array}$$

with variables $u \in \mathbf{R}^N$, $y \in \mathbf{R}^N$, and $t \in \mathbf{R}^N$, where

$$H = \left[\begin{array}{ccccc} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{array} \right].$$

There are several other possible LP formulations. For example, we can keep the state trajectory $x(0), \ldots, x(N)$ as optimization variables, and replace the equality constraint above, $Hu = x_{\rm des}$, with the equality constraints

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1, \qquad x(0) = 0, \qquad x(N) = x_{\rm des}.$$

In this formulation, the variables are $u \in \mathbf{R}^N$, $x(0), \ldots, x(N) \in \mathbf{R}^n$, as well as $y \in \mathbf{R}^N$ and $t \in \mathbf{R}^N$.

Yet another variation is to not use the intermediate variable y introduced above, and express the problem just in terms of the variable t (and u):

$$-t \preceq u \preceq t, \quad 2u - \mathbf{1} \preceq t, \quad -2u - \mathbf{1} \preceq t,$$

with variables $u \in \mathbf{R}^N$ and $t \in \mathbf{R}^N$.

4.30 A heated fluid at temperature T (degrees above ambient temperature) flows in a pipe with fixed length and circular cross section with radius r. A layer of insulation, with thickness $w \ll r$, surrounds the pipe to reduce heat loss through the pipe walls. The design variables in this problem are T, r, and w.

The heat loss is (approximately) proportional to Tr/w, so over a fixed lifetime, the energy cost due to heat loss is given by $\alpha_1 Tr/w$. The cost of the pipe, which has a fixed wall thickness, is approximately proportional to the total material, *i.e.*, it is given by $\alpha_2 r$. The cost of the insulation is also approximately proportional to the total insulation material, *i.e.*, $\alpha_3 rw$ (using $w \ll r$). The total cost is the sum of these three costs.

The heat flow down the pipe is entirely due to the flow of the fluid, which has a fixed velocity, *i.e.*, it is given by $\alpha_4 Tr^2$. The constants α_i are all positive, as are the variables T, r, and w.

Now the problem: maximize the total heat flow down the pipe, subject to an upper limit $C_{\max}$ on total cost, and the constraints

$$T_{\min} \leq T \leq T_{\max}, \qquad r_{\min} \leq r \leq r_{\max}, \qquad w_{\min} \leq w \leq w_{\max}, \quad w \leq 0.1r.$$

Express this problem as a geometric program.

Solution. The problem is

$$\begin{array}{ll} \text{maximize} & \alpha_4 Tr^2 \\ \text{subject to} & \alpha_1 Trw^{-1} + \alpha_2 r + \alpha_3 rw \leq C_{\max} \\ & T_{\min} \leq T \leq T_{\max} \\ & r_{\min} \leq r \leq r_{\max} \\ & w_{\min} \leq w \leq w_{\max} \\ & w \leq 0.1r. \end{array}$$

This is equivalent to the GP

$$\begin{array}{ll} \text{minimize} & (1/\alpha_4)T^{-1}r^{-2} \\ \text{subject to} & (\alpha_1/C_{\max})Trw^{-1} + (\alpha_2/C_{\max})r + (\alpha_3/C_{\max})rw \leq 1 \\ & (1/T_{\max})T \leq 1, \quad T_{\min}T^{-1} \leq 1 \\ & (1/r_{\max})r \leq 1, \quad r_{\min}r^{-1} \leq 1 \\ & (1/w_{\max})w \leq 1, \quad w_{\min}w^{-1} \leq 1 \\ & 10wr^{-1} \leq 1 \end{array}$$

(with variables T, r, w).

4.40 *LPs, QPs, QCQPs, and SOCPs as SDPs.* Express the following problems as SDPs.

(a) The LP (4.27).

Solution.

$$\begin{array}{ll}\text{minimize} & c^T x + d \\ \text{subject to} & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b.\end{array}$$

(b) The QP (4.34), the QCQP (4.35) and the SOCP (4.36). *Hint.* Suppose $A \in \mathbf{S}^r_{++}$, $C \in \mathbf{S}^s$, and $B \in \mathbf{R}^{r\times s}$. Then

$$\left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right] \succeq 0 \iff C - B^T A^{-1} B \succeq 0.$$

For a more complete statement, which applies also to singular A, and a proof, see §A.5.5.

Solution.

i. QP. Express $P = WW^T$ with $W \in \mathbf{R}^{n\times r}$.

$$\begin{array}{ll}\text{minimize} & t + 2q^T x + r \\ \text{subject to} & \left[\begin{array}{cc} I & W^T x \\ x^T W & tI \end{array}\right] \succeq 0 \\ & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b,\end{array}$$

with variables x, $t \in \mathbf{R}$.

ii. QCQP. Express $P_i = W_i W_i^T$ with $W_i \in \mathbf{R}^{n\times r_i}$.

$$\begin{array}{ll}\text{minimize} & t_0 + 2q_0^T x + r_0 \\ \text{subject to} & t_i + 2q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & \left[\begin{array}{cc} I & W_i^T x \\ x^T W_i & t_i I \end{array}\right] \succeq 0, \quad i = 0, 1, \ldots, m \\ & Ax = b,\end{array}$$

with variables x, $t_i \in \mathbf{R}$.

iii. SOCP.

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & \left[\begin{array}{cc} (c_i^T x + d_i)I & A_i x + b_i \\ (Ax_i + b_i)^T & (c_i^T x + d_i)I \end{array}\right] \succeq 0, \quad i = 1, \ldots, N \\ & Fx = g.\end{array}$$

By the result in the hint, the constraint is equivalent with $\|A_i x + b_i\|_2 < c_i^T x + d_i$ when $c_i^T x + d_i > 0$. We have to check the case $c_i^T x + d_i = 0$ separately. In this case, the LMI constraint means $A_i x + b_i = 0$, so we can conclude that the LMI constraint and the SOC constraint are equivalent.

(c) The matrix fractional optimization problem

$$\text{minimize} \quad (Ax+b)^T F(x)^{-1}(Ax+b)$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$,

$$F(x) = F_0 + x_1F_1 + \cdots + x_nF_n,$$

with $F_i \in \mathbf{S}^m$, and we take the domain of the objective to be $\{x \mid F(x) \succ 0\}$. You can assume the problem is feasible (there exists at least one x with $F(x) \succ 0$).

Solution.

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \end{array}$$

with variables x, $t \in \mathbf{R}$. The LMI constraint is equivalent to

$$(Ax+b)^T F(x)^{-1}(Ax+b) \le t$$

if $F(x) \succ 0$.

More generally, let

$$f_0(x) = (Ax+b)^T F(x)^{-1}(Ax+b), \qquad \mathbf{dom}\, f_0(x) = \{x \mid F(x) \succ 0\}.$$

We have

$$\mathbf{epi}\, f_0 = \left\{(x,t) \;\middle|\; F(x) \succ 0, \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \right\}.$$

Then $\mathbf{cl}(\mathbf{epi}\, f_0) = \mathbf{epi}\, g$ where g is defined by

$$\begin{array}{rcl} \mathbf{epi}\, g & = & \left\{(x,t) \;\middle|\; \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \right\} \\ g(x) & = & \inf \left\{ t \;\middle|\; \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & t \end{bmatrix} \succeq 0 \right\}. \end{array}$$

We conclude that both problems have the same optimal values. An optimal solution for the matrix fractional problem is optimal for the SDP. An optimal solution for the SDP, with $F(x) \succ 0$, is optimal for the matrix fractional problem. If $F(x)$ is singular at the optimal solution of the SDP, then the optimum for the matrix fractional problem is not attained.

5.1 *A simple example.* Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \le 0, \end{array}$$

with variable $x \in \mathbf{R}$.

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

(b) *Lagrangian and dual function.* Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?

(d) *Sensitivity analysis.* Let $p^\star(u)$ denote the optimal value of the problem

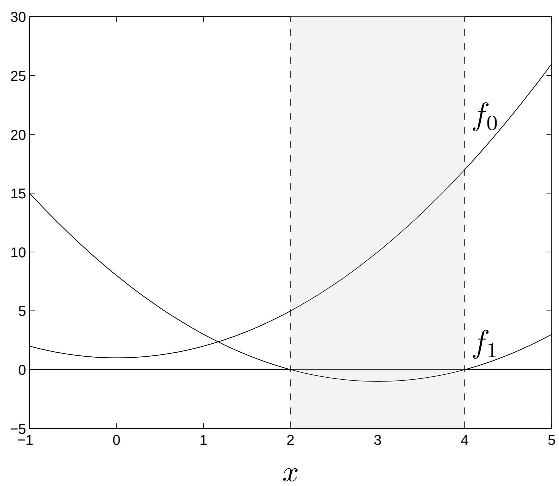
$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq u, \end{array}$$

as a function of the parameter u. Plot $p^\star(u)$. Verify that $dp^\star(0)/du = -\lambda^\star$.

Solution.

(a) The feasible set is the interval $[2, 4]$. The (unique) optimal point is $x^\star = 2$, and the optimal value is $p^\star = 5$.

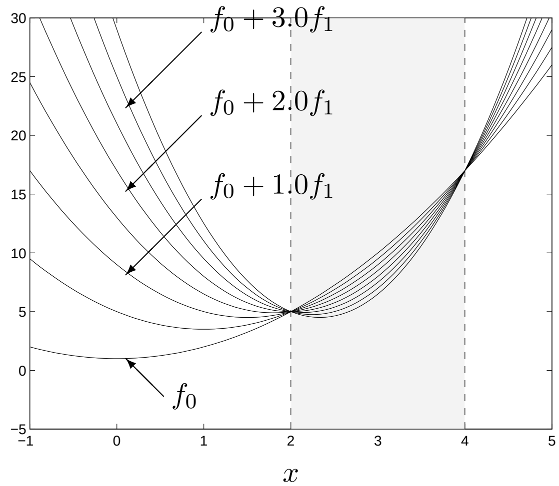
The plot shows f_0 and f_1.



(b) The Lagrangian is

$$L(x, \lambda) = (1 + \lambda)x^2 - 6\lambda x + (1 + 8\lambda).$$

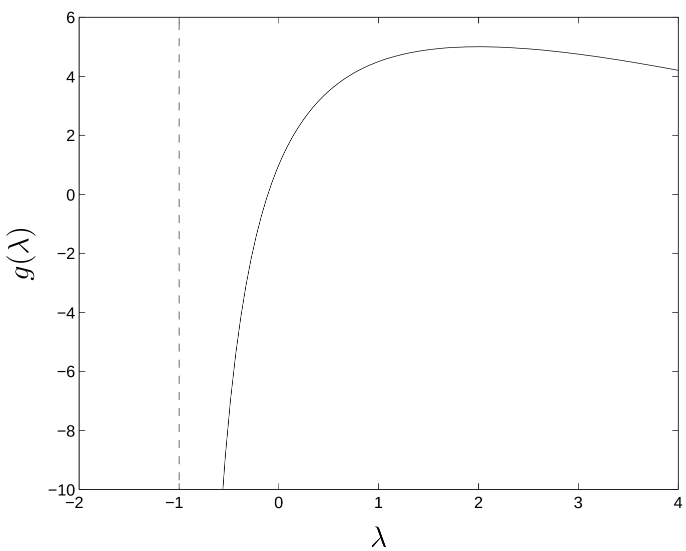
The plot shows the Lagrangian $L(x, \lambda) = f_0 + \lambda f_1$ as a function of x for different values of $\lambda \geq 0$. Note that the minimum value of $L(x, \lambda)$ over x (*i.e.*, $g(\lambda)$) is always less than $p^\star$. It increases as λ varies from 0 toward 2, reaches its maximum at $\lambda = 2$, and then decreases again as λ increases above 2. We have equality $p^\star = g(\lambda)$ for $\lambda = 2$.



For $\lambda > -1$, the Lagrangian reaches its minimum at $\tilde{x} = 3\lambda/(1+\lambda)$. For $\lambda \leq -1$ it is unbounded below. Thus

$$g(\lambda) = \begin{cases} -9\lambda^2/(1+\lambda) + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

which is plotted below.



We can verify that the dual function is concave, that its value is equal to $p^\star = 5$ for $\lambda = 2$, and less than $p^\star$ for other values of λ.

(c) The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda) + 1 + 8\lambda \\ \text{subject to} & \lambda \geq 0. \end{array}$$

The dual optimum occurs at $\lambda = 2$, with $d^\star = 5$. So for this example we can directly observe that strong duality holds (as it must — Slater's constraint qualification is satisfied).

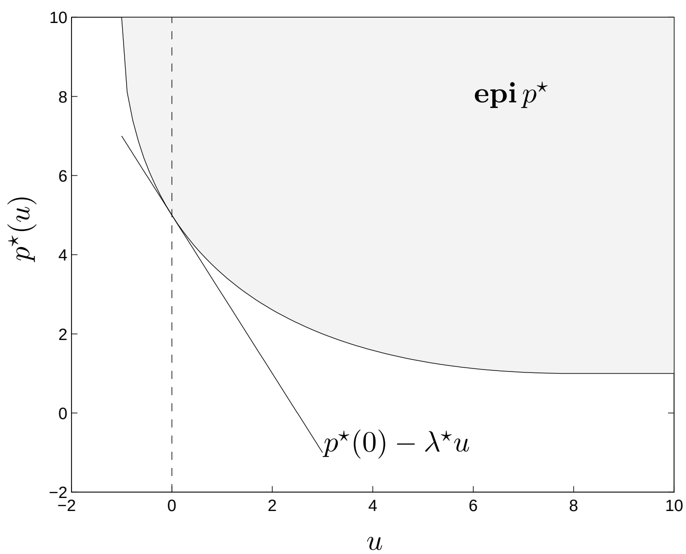
(d) The perturbed problem is infeasible for $u < -1$, since $\inf_x(x^2 - 6x + 8) = -1$. For $u \geq -1$, the feasible set is the interval

$$[3 - \sqrt{1+u}, 3 + \sqrt{1+u}],$$

given by the two roots of $x^2 - 6x + 8 = u$. For $-1 \leq u \leq 8$ the optimum is $x^\star(u) = 3 - \sqrt{1+u}$. For $u \geq 8$, the optimum is the unconstrained minimum of f_0, *i.e.*, $x^\star(u) = 0$. In summary,

$$p^\star(u) = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u \leq 8 \\ 1 & u \geq 8. \end{cases}$$

The figure shows the optimal value function $p^\star(u)$ and its epigraph.



Finally, we note that $p^\star(u)$ is a differentiable function of u, and that

$$\frac{dp^\star(0)}{du} = -2 = -\lambda^\star.$$

5.5 *Dual of general LP.* Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit.

Solution.

(a) The Lagrangian is

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x + \lambda^T(Gx - h) + \nu^T(Ax - b) \\ &= (c^T + \lambda^T G + \nu^T A)x - h\lambda^T - \nu^T b, \end{aligned}$$

which is an affine function of x. It follows that the dual function is given by

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -\lambda^T h - \nu^T b & c + G^T\lambda + A^T\nu = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(b) The dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

After making the implicit constraints explicit, we obtain

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \nu^T b \\ \text{subject to} & c + G^T\lambda + A^T\nu = 0 \\ & \lambda \succeq 0. \end{array}$$

Solutions to additional exercises

1. *Simple portfolio optimization.* We consider a portfolio optimization problem as described on pages 155 and 185–186 of *Convex Optimization*, with data that can be found in the file `simple_portfolio_data.m`.

   (a) Find minimum-risk portfolios with the same expected return as the uniform portfolio ($x = (1/n)\mathbf{1}$), with risk measured by portfolio return variance, and the following portfolio constraints (in addition to $\mathbf{1}^T x = 1$):

   - No (additional) constraints.
   - Long-only: $x \succeq 0$.
   - Limit on total short position: $\mathbf{1}^T(x_-) \leq 0.5$, where $(x_-)_i = \max\{-x_i, 0\}$.

   Compare the optimal risk in these portfolios with each other and the uniform portfolio.

   (b) Plot the optimal risk-return trade-off curves for the long-only portfolio, and for total short-position limited to 0.5, in the same figure. Follow the style of figure 4.12 (top), with horizontal axis showing standard deviation of portfolio return, and vertical axis showing mean return.

   Solution.

   (a) We can express these as QPs:

   - No (additional) constraints:

   $$\begin{array}{ll} \text{minimize} & x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad \overline{p}^T x = \overline{p}^T (1/n)\mathbf{1} \end{array}$$

   - Long only

   $$\begin{array}{ll} \text{minimize} & x^T \Sigma x \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1 \\ & \overline{p}^T x = \overline{p}^T (1/n)\mathbf{1} \end{array}$$

   - Limit on total short position:

   $$\begin{array}{ll} \text{minimize} & x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad \overline{p}^T x = \overline{p}^T (1/n)\mathbf{1} \\ & \mathbf{1}^T x_- \leq 0.5 \end{array}$$

   Although the two portfolios have the same expected return, their risk profiles differ drastically. The standard deviations are:

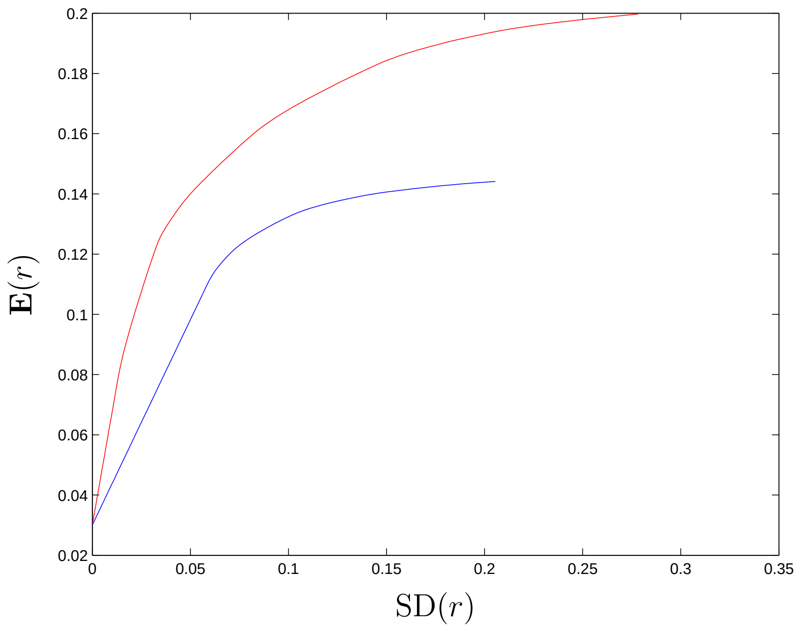
   - uniform: 8.7%
   - long-only: 5.1%
   - limit on total short position: 2.1%

- unconstrained: 1.9%.

Notice that as the size of the feasible set increases, the objective value improves.

(b) The optimal risk-return trade-off curves can be generated by scalarizing the bi-criterion objective, $\overline{p}^T x - \gamma x^T \Sigma x$, over a range of values for γ. For example, the optimal curve for the long-only portfolio is generated by solving the following family of QPs:

$$\begin{array}{ll} \text{maximize} & \overline{p}^T x - \gamma x^T \Sigma x \\ \text{subject to} & x \succeq 0, \quad \mathbf{1}^T x = 1 \end{array}$$



For every level of return, there is a portfolio with limits on total short positions (red) that has greater expected return than the optimal long-only portfolio (blue).

The following MATLAB script solves this problem:

```
simple_portfolio_data;
%% part i
%minimum-risk unconstrained portfolio with same expected return as uniform
%allocation
cvx_begin
cvx_quiet(true)
variable x_unconstrained(n)
minimize(quad_form(x_unconstrained,S))
subject to
    sum(x_unconstrained)==1;
    pbar'*x_unconstrained==x_unif'*pbar;
cvx_end
%% part ii
%minimum-risk long-only portfolio with same expected return as uniform
```

```
%allocation
cvx_begin
cvx_quiet(true)
variable x_long(n)
minimize(quad_form(x_long,S))
subject to
    x_long>=0;
    sum(x_long)==1;
    pbar'*x_long==x_unif'*pbar;
cvx_end
%% part iii
%minimum-risk constrained short portfolio with same expected return as uniform
%allocation
cvx_begin
cvx_quiet(true)
variable x_shortconstr(n)
minimize(quad_form(x_shortconstr,S))
subject to
    sum(pos(-x_shortconstr))<=0.5;
    sum(x_shortconstr)==1;
    pbar'*x_shortconstr==x_unif'*pbar;
cvx_end
%% Generate risk-return trade-off curves
sprintf('unconstrained sd: %0.3g\n', sqrt(quad_form(x_unconstrained,S)))
sprintf('long only sd: %0.3g\n', sqrt(quad_form(x_long,S)))
sprintf('constrained short sd: %0.3g\n', sqrt(quad_form(x_shortconstr,S)))
sprintf('x_unif sd: %0.3g\n', sqrt(quad_form(x_unif,S)))
novals=100;
r_long = [];
r_shortconstr = [];
sd_long = [];
sd_shortconstr = [];
muvals = logspace(-1,4,novals);

for i=1:novals
  mu  = muvals(i);
  %long only
  cvx_begin
  cvx_quiet(true)
  variable x(n)
  maximize(pbar'*x - mu*quad_form(x,S))
  subject to
```

```
      x>=0;
      sum(x)==1;
    cvx_end
    r_long = [r_long,  pbar'*x];
    sd_long = [sd_long,  sqrt(x'*S*x) ];
    %constrained short
    cvx_begin
    cvx_quiet(true)
    variables x(n)
    maximize(pbar'*x - mu*quad_form(x,S))
    subject to
      sum(x)==1;
      sum(pos(-x))<=.5;
    cvx_end
    r_shortconstr = [r_shortconstr, pbar'*x];
    sd_shortconstr = [sd_shortconstr, sqrt(x'*S*x)];
  end;

  plot(sd_long, r_long);
  hold; plot(sd_shortconstr, r_shortconstr, 'r');
```

2. *Minimum fuel optimal control.* Solve the minimum fuel optimal control problem described in exercise 4.16 of *Convex Optimization*, for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \quad x_{\mathrm{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad N = 30.$$

You can do this by forming the LP you found in your solution of exercise 4.16, or more directly using `cvx`. Plot the actuator signal $u(t)$ as a function of time t.

Solution. The following Matlab code finds the solution

```
close all
clear all

n=3;  % state dimension
N=30; % time horizon

A=[ -1 0.4 0.8; 1 0 0 ; 0 1 0];
b=[ 1 0 0.3]';
x0 = zeros(n,1);
xdes = [ 7 2 -6]';
```

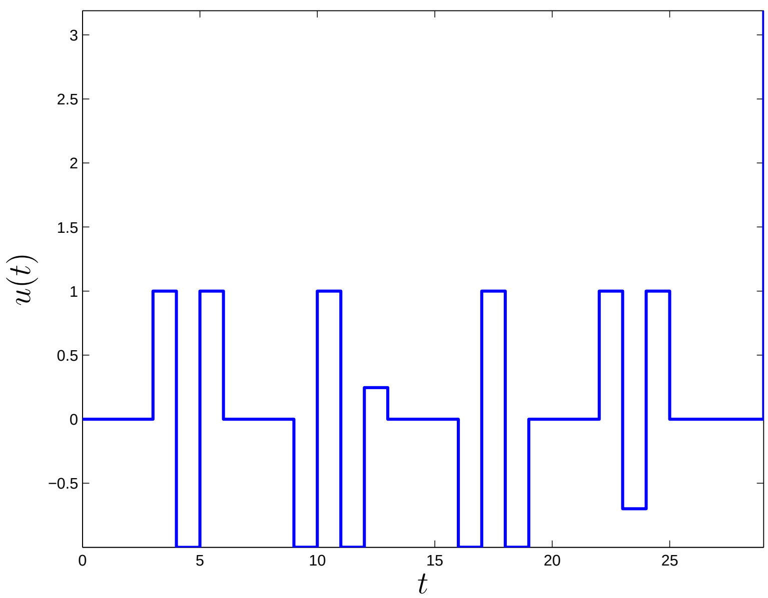


Figure 1 Minimum fuel actuator signal.

```
cvx_begin
    variable X(n,N+1);
    variable u(1,N);
    minimize (sum(max(abs(u),2*abs(u)-1)))
    subject to
        X(:,2:N+1) == A*X(:,1:N)+b*u;  % dynamics
        X(:,1) == x0;
        X(:,N+1) == xdes;
cvx_end

stairs(0:N-1,u,'linewidth',2)
axis tight
xlabel('t')
ylabel('u')
```

The optimal actuator signal is shown in figure 1.

3. *Numerical perturbation analysis example.* Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} & x_1 + 2x_2 \leq u_1 \\ & x_1 - 4x_2 \leq u_2, \\ & 5x_1 + 76x_2 \leq 1, \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

(a) Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values $x_1^\star$ and $x_2^\star$, and optimal dual variable values $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$. Let $p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Hint: See §3.6 of the `CVX` users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use `quad_form()`.

(b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \qquad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction $p^\star_{\rm pred}$ of the optimal value of the perturbed QP, and compare it to $p^\star_{\rm exact}$, the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p^\star_{\rm pred} \leq p^\star_{\rm exact}$ holds.

δ_1	δ_2	$p^\star_{\rm pred}$	$p^\star_{\rm exact}$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

Solution.

(a) The following Matlab code sets up the simple QP and solves it using `CVX`:

```
Q = [1 -1/2; -1/2 2];
f = [-1 0]';
A = [1 2; 1 -4; 5 76];
b = [-2 -3 1]';

cvx_begin
    variable x(2)
    dual variable lambda
    minimize(quad_form(x,Q)+f'*x)
    subject to
```

```
        lambda: A*x <= b
cvx_end
p_star = cvx_optval
```

When we run this, we find the optimal objective value is $p^\star = 8.22$ and the optimal point is $x_1^\star = -2.33$, $x_2^\star = 0.17$. (This optimal point is unique since the objective is strictly convex.) A set of optimal dual variables is $\lambda_1^\star = 2.13$, $\lambda_2^\star = 3.31$ and $\lambda_3^\star = 0.08$. (The dual optimal point is unique too, but it's harder to show this, and it doesn't matter anyway.)

The KKT conditions are

$$
\begin{array}{lll}
x_1^\star + 2x_2^\star \leq u_1, & x_1^\star - 4x_2^\star \leq u_2, & 5x_1^\star + 76x_2^\star \leq 1 \\
\lambda_1^\star \geq 0, & \lambda_2^\star \geq 0, & \lambda_3^\star \geq 0 \\
\lambda_1^\star(x_1^\star + 2x_2^\star - u_1) = 0, & \lambda_2^\star(x_1^\star - 4x_2^\star - u_2) = 0, & \lambda_3^\star(5x_1^\star + 76x_2^\star - 1) = 0, \\
2x_1^\star - x_2^\star - 1 + \lambda_1^\star + \lambda_2^\star + 5\lambda_3^\star = 0, & & \\
4x_2^\star - x_1^\star + 2\lambda_1^\star - 4\lambda_2^\star + 76\lambda_3^\star = 0. & &
\end{array}
$$

We check these numerically. The dual variable $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$ are all greater than zero and the quantities

```
A*x-b
2*Q*x+f+A'*lambda
```

are found to be very small. Thus the KKT conditions are verified.

(b) The predicted optimal value is given by

$$p^\star_{\text{pred}} = p^\star - \lambda_1^\star \delta_1 - \lambda_2^\star \delta_2.$$

The following Matlab code fills in the table

```
arr_i = [0 -1 1];
delta = 0.1;
pa_table = [];
for i = arr_i
    for j = arr_i
        p_pred = p_star - [lambda(1) lambda(2)]*[i; j]*delta;
        cvx_begin
            variable x(2)
            minimize(quad_form(x,Q)+f'*x)
            subject to
                A*x <= b+[i;j;0]*delta
        cvx_end
        p_exact = cvx_optval;

        pa_table = [pa_table; i*delta j*delta p_pred p_exact]
    end
end
```

The values obtained are

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0	8.22	8.22
0	-0.1	8.55	8.70
0	0.1	7.89	7.98
-0.1	0	8.44	8.57
-0.1	-0.1	8.77	8.82
-0.1	0.1	8.10	8.32
0.1	0	8.01	8.22
0.1	-0.1	8.34	8.71
0.1	0.1	7.68	7.75

The inequality $p^\star_{\text{pred}} \leq p^\star_{\text{exact}}$ is verified to be true in all cases.