Lecture 13
Linear quadratic Lyapunov theory

- the Lyapunov equation
- Lyapunov stability conditions
- the Lyapunov operator and integral
- evaluating quadratic integrals
- analysis of ARE
- discrete-time results
- linearization theorem

The Lyapunov equation

the *Lyapunov equation* is

$$A^T P + PA + Q = 0$$

where $A,\ P,\ Q \in \mathbf{R}^{n\times n}$, and $P,\ Q$ are symmetric

interpretation: for linear system $\dot{x} = Ax$, if $V(z) = z^T P z$, then

$$\dot{V}(z) = (Az)^T P z + z^T P (Az) = -z^T Q z$$

$i.e.$, if $z^T P z$ is the (generalized)*energy*, then $z^T Q z$ is the associated (generalized) *dissipation*

linear-quadratic Lyapunov theory: *linear* dynamics, *quadratic* Lyapunov function we consider system $\dot{x} = Ax$, with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A

if $P > 0$, then

- the sublevel sets are ellipsoids (and bounded)
- $V(z) = z^T P z = 0 \;\Leftrightarrow\; z = 0$

boundedness condition: if $P > 0$, $Q \geq 0$ then

- all trajectories of $\dot{x} = Ax$ are bounded
  (this means $\Re\lambda_i \leq 0$, and if $\Re\lambda_i = 0$, then λ_i corresponds to a Jordan block of size one)
- the ellipsoids $\{z \mid z^T P z \leq a\}$ are invariant

Stability condition

if $P > 0$, $Q > 0$ then the system $\dot{x} = Ax$ is (globally asymptotically) stable, *i.e.*, $\Re\lambda_i < 0$

to see this, note that

$$\dot{V}(z) = -z^T Q z \leq -\lambda_{\min}(Q) z^T z \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} z^T P z = -\alpha V(z)$$

where $\alpha = \lambda_{\min}(Q)/\lambda_{\max}(P) > 0$

An extension based on observability

(Lasalle's theorem for linear dynamics, quadratic function)

if $P > 0$, $Q \geq 0$, and (Q, A) observable, then the system $\dot{x} = Ax$ is (globally asymptotically) stable

to see this, we first note that all eigenvalues satisfy $\Re\lambda_i \leq 0$

now suppose that $v \neq 0$, $Av = \lambda v$, $\Re\lambda = 0$

then $A\bar{v} = \bar{\lambda}\bar{v} = -\lambda\bar{v}$, so

$$\left\| Q^{1/2} v \right\|^2 = v^* Q v = -v^* \left(A^T P + PA \right) v = \lambda v^* P v - \lambda v^* P v = 0$$

which implies $Q^{1/2}v = 0$, so $Qv = 0$, contradicting observability (by PBH)

interpretation: observability condition means no trajectory can stay in the "zero dissipation" set $\{z \mid z^T Q z = 0\}$

An instability condition

if $Q \geq 0$ and $P \not\geq 0$, then A is not stable

to see this, note that $\dot{V} \leq 0$, so $V(x(t)) \leq V(x(0))$

since $P \not\geq 0$, there is a w with $V(w) < 0$; trajectory starting at w does not converge to zero

in this case, the sublevel sets $\{z \mid V(z) \leq 0\}$ (which are unbounded) are invariant

The Lyapunov operator

the *Lyapunov operator* is given by

$$\mathcal{L}(P) = A^T P + PA$$

special case of Sylvester operator

$\mathcal{L}$ is nonsingular if and only if A and $-A$ share no common eigenvalues, $i.e.$, A does not have pair of eigenvalues which are negatives of each other

- if A is stable, Lyapunov operator is nonsingular
- if A has imaginary (nonzero, $i\omega$-axis) eigenvalue, then Lyapunov operator is singular

thus if A is stable, for any Q there is exactly one solution P of Lyapunov equation $A^T P + PA + Q = 0$

Solving the Lyapunov equation

$$A^T P + PA + Q = 0$$

we are given A and Q and want to find P

if Lyapunov equation is solved as a set of $n(n+1)/2$ equations in $n(n+1)/2$ variables, cost is $O(n^6)$ operations

fast methods, that exploit the special structure of the linear equations, can solve Lyapunov equation with cost $O(n^3)$

based on first reducing A to Schur or upper Hessenberg form

The Lyapunov integral

if A is stable there is an explicit formula for solution of Lyapunov equation:

$$P = \int_0^\infty e^{tA^T} Q e^{tA}\, dt$$

to see this, we note that

$$\begin{aligned} A^T P + PA &= \int_0^\infty \left(A^T e^{tA^T} Q e^{tA} + e^{tA^T} Q e^{tA} A \right) dt \\ &= \int_0^\infty \left(\frac{d}{dt} e^{tA^T} Q e^{tA} \right) dt \\ &= e^{tA^T} Q e^{tA} \Big|_0^\infty \\ &= -Q \end{aligned}$$

Interpretation as cost-to-go

if A is stable, and P is (unique) solution of $A^TP + PA + Q = 0$, then

$$
\begin{aligned}
V(z) &= z^TPz \\
&= z^T\left(\int_0^\infty e^{tA^T}Qe^{tA}\,dt\right)z \\
&= \int_0^\infty x(t)^TQx(t)\,dt
\end{aligned}
$$

where $\dot{x} = Ax$, $x(0) = z$

thus $V(z)$ is cost-to-go from point z (with no input) and integral quadratic cost function with matrix Q

if A is stable and $Q > 0$, then for each t, $e^{tA^T} Q e^{tA} > 0$, so

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \, dt > 0$$

meaning: if A is stable,

- we can choose *any* positive definite quadratic form $z^T Q z$ as the dissipation, *i.e.*, $-\dot{V} = z^T Q z$
- then solve a set of linear equations to find the (unique) quadratic form $V(z) = z^T P z$
- V will be positive definite, so it is a Lyapunov function that proves A is stable

in particular: *a linear system is stable if and only if there is a quadratic Lyapunov function that proves it*

generalization: if A stable, $Q \geq 0$, and (Q, A) observable, then $P > 0$

to see this, the Lyapunov integral shows $P \geq 0$

if $Pz = 0$, then

$$0 = z^T P z = z^T \left(\int_0^\infty e^{tA^T} Q e^{tA} \, dt \right) z = \int_0^\infty \left\| Q^{1/2} e^{tA} z \right\|^2 \, dt$$

so we conclude $Q^{1/2} e^{tA} z = 0$ for all $t \geq 0$

this implies that $Qz = 0$, $QAz = 0$, . . . , $QA^{n-1}z = 0$, contradicting (Q, A) observable

Monotonicity of Lyapunov operator inverse

suppose $A^T P_i + P_i A + Q_i = 0$, $i = 1,\ 2$

if $Q_1 \geq Q_2$, then for all t, $e^{tA^T} Q_1 e^{tA} \geq e^{tA^T} Q_2 e^{tA}$

if A is stable, we have

$$P_1 = \int_0^\infty e^{tA^T} Q_1 e^{tA}\, dt \geq \int_0^\infty e^{tA^T} Q_2 e^{tA}\, dt = P_2$$

in other words: if A is stable then

$$Q_1 \geq Q_2 \implies \mathcal{L}^{-1}(Q_1) \geq \mathcal{L}^{-1}(Q_2)$$

$i.e.$, inverse Lyapunov operator is monotonic, or preserves matrix inequality, when A is stable

(question: is $\mathcal{L}$ monotonic?)

Evaluating quadratic integrals

suppose $\dot{x} = Ax$ is stable, and define

$$J = \int_0^\infty x(t)^T Q x(t)\, dt$$

to find J, we solve Lyapunov equation $A^T P + PA + Q = 0$ for P

then, $J = x(0)^T P x(0)$

in other words: we can evaluate quadratic integral exactly, by solving a set of linear equations, without even computing a matrix exponential

Controllability and observability Grammians

for A stable, the controllability Grammian of (A, B) is defined as

$$W_c = \int_0^\infty e^{tA} B B^T e^{tA^T} \, dt$$

and the observability Grammian of (C, A) is

$$W_o = \int_0^\infty e^{tA^T} C^T C e^{tA} \, dt$$

these Grammians can be computed by solving the Lyapunov equations

$$A W_c + W_c A^T + B B^T = 0, \qquad A^T W_o + W_o A + C^T C = 0$$

we always have $W_c \geq 0$, $W_o \geq 0$;
$W_c > 0$ if and only if (A, B) is controllable, and
$W_o > 0$ if and only if (C, A) is observable

Evaluating a state feedback gain

consider

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad u = Kx, \qquad x(0) = x_0$$

with closed-loop system $\dot{x} = (A + BK)x$ stable

to evaluate the quadratic integral performance measures

$$J_u = \int_0^\infty u(t)^T u(t)\, dt, \qquad J_y = \int_0^\infty y(t)^T y(t)\, dt$$

we solve Lyapunov equations

$$\begin{aligned}(A + BK)^T P_u + P_u(A + BK) + K^T K &= 0 \\ (A + BK)^T P_y + P_y(A + BK) + C^T C &= 0\end{aligned}$$

then we have $J_u = x_0^T P_u x_0$, $J_y = x_0^T P_y x_0$

Lyapunov analysis of ARE

write ARE (with $Q \geq 0$, $R > 0$)

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

as

$$(A + BK)^T P + P(A + BK) + (Q + K^T RK) = 0$$

with $K = -R^{-1}B^T P$

we conclude: if $A + BK$ stable, then $P \geq 0$ (since $Q + K^T RK \geq 0$)

i.e., any stabilizing solution of ARE is PSD

if also (Q, A) is observable, then we conclude $P > 0$

to see this, we need to show that $(Q + K^T RK, A + BK)$ is observable

if not, there is $v \neq 0$ s.t.

$$(A + BK)v = \lambda v, \qquad (Q + K^T RK)v = 0$$

which implies

$$v^*(Q + K^T RK)v = v^*Qv + v^*K^T RKv = \|Q^{1/2}v\|^2 + \|R^{1/2}Kv\|^2 = 0$$

so $Qv = 0$, $Kv = 0$

$$(A + BK)v = Av = \lambda v, \qquad Qv = 0$$

which contradicts (Q, A) observable

the same argument shows that if $P > 0$ and (Q, A) is observable, then $A + BK$ is stable

Monotonic norm convergence

suppose that $A + A^T < 0$, $i.e.$, (symmetric part of) A is negative definite

can express as $A^T P + PA + Q = 0$, with $P = I$, $Q > 0$

meaning: $x^T P x = \|x\|^2$ decreases along every nonzero trajectory, $i.e.$,

- $\|x(t)\|$ is always *decreasing monotonically* to 0
- $x(t)$ is always moving towards origin

this implies A is stable, but the converse is false: for a stable system, we need not have $A + A^T < 0$

(for a stable system with $A + A^T \not< 0$, $\|x(t)\|$ converges to zero, but not monotonically) for a stable system we can always change coordinates so we have monotonic norm convergence

let P denote the solution of $A^T P + PA + I = 0$

take $T = P^{-1/2}$

in new coordinates A becomes $\tilde{A} = T^{-1}AT$,

$$\begin{aligned} \tilde{A} + \tilde{A}^T &= P^{1/2}AP^{-1/2} + P^{-1/2}A^T P^{1/2} \\ &= P^{-1/2}\left(PA + A^T P\right)P^{-1/2} \\ &= -P^{-1} < 0 \end{aligned}$$

in new coordinates, convergence is *obvious* because $\|x(t)\|$ is always decreasing

Discrete-time results

all linear quadratic Lyapunov results have discrete-time counterparts

the *discrete-time* Lyapunov equation is

$$A^T PA - P + Q = 0$$

meaning: if $x_{t+1} = Ax_t$ and $V(z) = z^T Pz$, then $\Delta V(z) = -z^T Qz$

- if $P > 0$ and $Q > 0$, then A is (discrete-time) stable ($i.e.$, $|\lambda_i| < 1$)
- if $P > 0$ and $Q \geq 0$, then all trajectories are bounded ($i.e.$, $|\lambda_i| \leq 1$; $|\lambda_i| = 1$ only for 1×1 Jordan blocks)
- if $P > 0$, $Q \geq 0$, and (Q, A) observable, then A is stable
- if $P \not> 0$ and $Q \geq 0$, then A is not stable

Discrete-time Lyapunov operator

the discrete-time Lyapunov operator is given by $\mathcal{L}(P) = A^T PA - P$

$\mathcal{L}$ is nonsingular if and only if, for all $i,\ j$, $\lambda_i \lambda_j \neq 1$
(roughly speaking, if and only if A and A^{-1} share no eigenvalues)

if A is stable, then $\mathcal{L}$ is nonsingular; in fact

$$P = \sum_{t=0}^{\infty} (A^T)^t Q A^t$$

is the unique solution of Lyapunov equation $A^T PA - P + Q = 0$

the discrete-time Lyapunov equation can be solved quickly (*i.e.*, $O(n^3)$)
and can be used to evaluate infinite sums of quadratic functions, etc.

Converse theorems

suppose $x_{t+1} = Ax_t$ is stable, $A^T PA - P + Q = 0$

- if $Q > 0$ then $P > 0$
- if $Q \geq 0$ and (Q, A) observable, then $P > 0$

in particular, a discrete-time linear system is stable if and only if there is a quadratic Lyapunov function that proves it

Monotonic norm convergence

suppose $A^T PA - P + Q = 0$, with $P = I$ and $Q > 0$

this means $A^T A < I$, $i.e.$, $\|A\| < 1$

meaning: $\|x_t\|$ decreases on every nonzero trajectory; indeed, $\|x_{t+1}\| \leq \|A\|\|x_t\| < \|x_t\|$

when $\|A\| < 1$,

- stability is obvious, since $\|x_t\| \leq \|A\|^t \|x_0\|$
- system is called *contractive* since norm is reduced at each step

the converse is false: system can be stable without $\|A\| < 1$

now suppose A is stable, and let P satisfy $A^TPA - P + I = 0$

take $T = P^{-1/2}$

in new coordinates A becomes $\tilde{A} = T^{-1}AT$, so

$$\begin{aligned} \tilde{A}^T\tilde{A} &= P^{-1/2}A^TPAP^{-1/2} \\ &= P^{-1/2}(P-I)P^{-1/2} \\ &= I - P^{-1} < I \end{aligned}$$

$i.e.$, $\|\tilde{A}\| < 1$

so for a stable system, we can change coordinates so the system is contractive

Lyapunov's linearization theorem

we consider nonlinear time-invariant system $\dot{x} = f(x)$, where $f : \mathbf{R}^n \to \mathbf{R}^n$

suppose x_e is an equilibrium point, *i.e.*, $f(x_e) = 0$, and let $A = Df(x_e) \in \mathbf{R}^{n \times n}$

the linearized system, near x_e, is $\dot{\delta x} = A\delta x$

linearization theorem:

- if the linearized system is stable, *i.e.*, $\Re\lambda_i(A) < 0$ for $i = 1, \ldots, n$, then the nonlinear system is locally asymptotically stable
- if for some i, $\Re\lambda_i(A) > 0$, then the nonlinear system is not locally asymptotically stable

stability of the linearized system determines the local stability of the nonlinear system, *except* when all eigenvalues are in the closed left halfplane, and at least one is on the imaginary axis

examples like $\dot{x} = x^3$ (which is not LAS) and $\dot{x} = -x^3$ (which is LAS) show the theorem cannot, in general, be tightened

examples:

eigenvalues of $Df(x_e)$	conclusion about $\dot{x} = f(x)$
$-3, \ -0.1 \pm 4i, \ -0.2 \pm i$	LAS near x_e
$-3, \ -0.1 \pm 4i, \ 0.2 \pm i$	not LAS near x_e
$-3, \ -0.1 \pm 4i, \ \pm i$	no conclusion

Proof of linearization theorem

let's assume $x_e = 0$, and express the nonlinear differential equation as

$$\dot{x} = Ax + g(x)$$

where $\|g(x)\| \leq K\|x\|^2$

suppose that A is stable, and let P be unique solution of Lyapunov equation

$$A^T P + PA + I = 0$$

the Lyapunov function $V(z) = z^T Pz$ proves stability of the linearized system; we'll use it to prove local asymptotic stability of the nonlinear system

$$\begin{aligned}
\dot{V}(z) &= 2z^TP(Az + g(z)) \\
&= z^T(A^TP + PA)z + 2z^TPg(z) \\
&= -z^Tz + 2z^TPg(z) \\
&\leq -\|z\|^2 + 2\|z\|\|P\|\|g(z)\| \\
&\leq -\|z\|^2 + 2K\|P\|\|z\|^3 \\
&= -\|z\|^2(1 - 2K\|P\|\|z\|)
\end{aligned}$$

so for $\|z\| \leq 1/(4K\|P\|)$,

$$\dot{V}(z) \leq -\frac{1}{2}\|z\|^2 \leq -\frac{1}{2\lambda_{\max}(P)}z^TPz = -\frac{1}{2\|P\|}z^TPz$$

finally, using

$$\|z\|^2 \leq \frac{1}{\lambda_{\min}(P)} z^T P z$$

we have

$$V(z) \leq \frac{\lambda_{\min}(P)}{16K^2\|P\|^2} \implies \|z\| \leq \frac{1}{4K\|P\|} \implies \dot{V}(z) \leq -\frac{1}{2\|P\|} V(z)$$

and we're done

comments:

- proof actually constructs an ellipsoid inside basin of attraction of $x_e = 0$, and a bound on exponential rate of convergence
- choice of $Q = I$ was arbitrary; can get better estimates using other Qs, better bounds on g, tighter bounding arguments . . .

Integral quadratic performance

consider $\dot{x} = f(x)$, $x(0) = x_0$

we are interested in the integral quadratic performance measure

$$J(x_0) = \int_0^\infty x(t)^T Q x(t) \, dt$$

for any fixed x_0 we can find this (approximately) by simulation and numerical integration

(we'll assume the integral exists; we do not require $Q \geq 0$)

Lyapunov bounds on integral quadratic performance

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z
- $\dot{V}(z) \leq -z^T Q z$ for all z

then we have $J(x_0) \leq V(x_0)$, *i.e.*, the Lyapunov function V serves as an upper bound on the integral quadratic cost

(since Q need not be PSD, we might not have $\dot{V} \leq 0$; so we cannot conclude that trajectories are bounded) to show this, we note that

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}(x(t))\, dt \leq -\int_0^T x(t)^T Q x(t)\, dt$$

and so

$$\int_0^T x(t)^T Q x(t)\, dt \leq V(x(0)) - V(x(T)) \leq V(x(0))$$

since this holds for arbitrary T, we conclude

$$\int_0^\infty x(t)^T Q x(t)\, dt \leq V(x(0))$$

Integral quadratic performance for linear systems

for a stable linear system, with $Q \geq 0$, the Lyapunov bound is sharp, *i.e.*, there exists a V such that

- $V(z) \geq 0$ for all z
- $\dot{V}(z) \leq -z^T Q z$ for all z

and for which $V(x_0) = J(x_0)$ for all x_0

(take $V(z) = z^T P z$, where P is solution of $A^T P + PA + Q = 0$)