Lecture 15
Linear matrix inequalities and the S-procedure

- Linear matrix inequalities
- Semidefinite programming
- S-procedure for quadratic forms and quadratic functions

Linear matrix inequalities

suppose $F_0, \ldots, F_n$ are symmetric $m \times m$ matrices

an inequality of the form

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0$$

is called a *linear matrix inequality* (LMI) in the variable $x \in \mathbf{R}^n$

here, $F : \mathbf{R}^n \to \mathbf{R}^{m \times m}$ is an affine function of the variable x

LMIs:

- can represent a wide variety of inequalities
- arise in many problems in control, signal processing, communications, statistics, . . .

most important for us: **LMIs can be solved very efficiently** by newly developed methods (EE364)

“solved” means: we can find x that satisfies the LMI, or determine that no solution exists

Example

$$F(x) = \begin{bmatrix} x_1 + x_2 & x_2 + 1 \\ x_2 + 1 & x_3 \end{bmatrix} \geq 0$$

$$F_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

LMI $F(x) \geq 0$ equivalent to

$$x_1 + x_2 \geq 0, \qquad x_3 \geq 0$$

$$(x_1 + x_2)x_3 - (x_2 + 1)^2 = x_1 x_3 + x_2 x_3 - x_2^2 - 2x_2 - 1 \geq 0$$

. . . a set of *nonlinear* inequalities in x

Certifying infeasibility of an LMI

- if A, B are symmetric PSD, then $\mathbf{Tr}(AB) \geq 0$:

$$\mathbf{Tr}(AB) = \mathbf{Tr}\left(A^{1/2}B^{1/2}B^{1/2}A^{1/2}\right) = \left\|A^{1/2}B^{1/2}\right\|_F^2$$

- suppose $Z = Z^T$ satisfies

$$Z \geq 0, \qquad \mathbf{Tr}(F_0 Z) < 0, \qquad \mathbf{Tr}(F_i Z) = 0, \quad i = 1, \ldots, n$$

- then if $F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0$,

$$0 \leq \mathbf{Tr}(ZF(x)) = \mathbf{Tr}(ZF_0) < 0$$

  a contradiction

- Z is *certificate* that proves LMI $F(x) \geq 0$ is infeasible

Example: Lyapunov inequality

suppose $A \in \mathbf{R}^{n \times n}$

the *Lyapunov inequality* $A^T P + PA + Q \leq 0$ is an LMI in variable P

meaning: P satisfies the Lyapunov LMI if and only if the quadratic form $V(z) = z^T P z$ satisfies $\dot{V}(z) \leq -z^T Q z$, for system $\dot{x} = Ax$

the dimension of the variable P is $n(n+1)/2$ (since $P = P^T$)

here, $F(P) = -A^T P - PA - Q$ is affine in P

(we don't need special LMI methods to solve the Lyapunov inequality; we can solve it analytically by solving the Lyapunov equation $A^T P + PA + Q = 0$)

Extensions

multiple LMIs: we can consider multiple LMIs as one, large LMI, by forming block diagonal matrices:

$$F^{(1)}(x) \geq 0, \ldots, F^{(k)}(x) \geq 0 \iff \mathbf{diag}\left(F^{(1)}(x), \ldots, F^{(k)}(x)\right) \geq 0$$

example: we can express a set of linear inequalities as an LMI with diagonal matrices:

$$a_1^T x \leq b_1, \ldots, a_k^T x \leq b_k \iff \mathbf{diag}(b_1 - a_1^T x, \ldots, b_k - a_k^T x) \geq 0$$

linear equality constraints: $a^T x = b$ is the same as the pair of linear inequalities $a^T x \leq b$, $a^T x \geq b$

Example: bounded-real LMI

suppose $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, and $\gamma > 0$

the *bounded-real LMI* is

$$\left[\begin{array}{cc} A^TP + PA + C^TC & PB \\ B^TP & -\gamma^2 I \end{array}\right] \leq 0, \qquad P \geq 0$$

with variable P

meaning: if P satisfies this LMI, then the quadratic Lyapunov function $V(z) = z^TPz$ proves the RMS gain of the system $\dot{x} = Ax + Bu$, $y = Cx$ is no more than γ

(in fact we can solve this LMI by solving an ARE-like equation, so we don't need special LMI methods . . .)

Strict inequalities in LMIs

sometimes we encounter strict matrix inequalities

$$F(x) \geq 0, \qquad F_{\text{strict}}(x) > 0$$

where F, F_{strict} are affine functions of x

- practical approach: replace $F_{\text{strict}}(x) > 0$ with $F_{\text{strict}}(x) \geq \epsilon I$, where ϵ is small and positive
- if F and F_{strict} are homogenous ($i.e.$, linear functions of x) we can replace with

$$F(x) \geq 0, \qquad F_{\text{strict}}(x) \geq I$$

example: we can replace $A^T P + PA \leq 0$, $P > 0$ (with variable P) with $A^T P + PA \leq 0$, $P \geq I$

Quadratic Lyapunov function for time-varying LDS

we consider time-varying linear system $\dot{x}(t) = A(t)x(t)$ with

$$A(t) \in \{A_1, \ldots, A_K\}$$

- we want to establish some property, such as all trajectories are bounded
- this is hard to do in general (cf. time-invariant LDS)
- let's use quadratic Lyapunov function $V(z) = z^T Pz$; we need $P > 0$, and $\dot{V}(z) \leq 0$ for all z, and all possible values of $A(t)$
- gives

$$P > 0, \qquad A_i^T P + PA_i \leq 0, \quad i = 1, \ldots, K$$

- by homogeneity, can write as LMIs

$$P \geq I, \qquad A_i^T P + PA_i \leq 0, \quad i = 1, \ldots, K$$

- in this case V is called *simultaneous Lyapunov function* for the systems $\dot{x} = A_i x$, $i = 1, \ldots, K$

- there is no analytical method ($e.g.$, using AREs) to solve such an LMI, but it is easily done numerically

- if such a P exists, it proves boundedness of trajectories of $\dot{x}(t) = A(t)x(t)$, with

$$A(t) = \theta_1(t)A_1 + \cdots + \theta_K(t)A_K$$

  where $\theta_i(t) \geq 0$

- in fact, it works for the *nonlinear* system $\dot{x} = f(x)$ provided for each $z \in \mathbf{R}^n$,

$$Df(z) = \theta_1(z)A_1 + \cdots + \theta_K(z)A_K$$

  for some $\theta_i(z) \geq 0$, $\theta_1(z) + \cdots + \theta_K(z) = 1$

Semidefinite programming

a *semidefinite program* (SDP) is an optimization problem with linear objective and LMI and linear equality constraints:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0 \\
& Ax = b
\end{array}
$$

most important property for us:
we can solve SDPs globally and efficiently

meaning: we either find a globally optimal solution, or determine that there is no x that satisfies the LMI & equality constraints *example:* let $A \in \mathbf{R}^{n \times n}$ be stable, $Q = Q^T \geq 0$

then the LMI $A^T P + PA + Q \leq 0$, $P \geq 0$ in P means the quadratic Lyapunov function $V(z) = z^T P z$ proves the bound

$$\int_0^\infty x(t)^T Q x(t)\, dt \leq x(0)^T P x(0)$$

now suppose that $x(0)$ is fixed, and we seek the best possible such bound

this can be found by solving the SDP

$$\begin{array}{ll} \text{minimize} & x(0)^T P x(0) \\ \text{subject to} & A^T P + PA + Q \leq 0, \quad P \geq 0 \end{array}$$

with variable P (note that the objective is linear in P)

(in fact we can solve this SDP analytically, by solving the Lyapunov equation)

S-procedure for two quadratic forms

let $F_0 = F_0^T$, $F_1 = F_1^T \in \mathbf{R}^{n \times n}$

when is it true that, for all z, $z^T F_1 z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$?

in other words, when does nonnegativity of one quadratic form imply nonnegativity of another?

simple condition: there exists $\tau \in \mathbf{R}$, $\tau \geq 0$, with $F_0 \geq \tau F_1$

then for sure we have $z^T F_1 z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$

(since if $z^T F_1 z \geq 0$, we then have $z^T F_0 z \geq \tau z^T F_1 z \geq 0$)

fact: the converse holds, provided there exists a point u with $u^T F_1 u > 0$

this result is called the *lossless* S-procedure, and is *not* easy to prove

(condition that there exists a point u with $u^T F_1 u > 0$ is called a *constraint qualification*)

S-procedure with strict inequalities

when is it true that, for all z, $z^T F_1 z \geq 0, \; z \neq 0 \; \Rightarrow \; z^T F_0 z > 0$?

in other words, when does nonnegativity of one quadratic form imply positivity of another for nonzero z?

simple condition: suppose there is a $\tau \in \mathbf{R}$, $\tau \geq 0$, with $F_0 > \tau F_1$

fact: the converse holds, provided there exists a point u with $u^T F_1 u > 0$

again, this is *not* easy to prove

Example

let's use quadratic Lyapunov function $V(z) = z^T P z$ to prove stability of

$$\dot{x} = Ax + g(x), \qquad \|g(x)\| \leq \gamma \|x\|$$

we need $P > 0$ and $\dot{V}(x) \leq -\alpha V(x)$ for all x ($\alpha > 0$ is given)

$$\begin{aligned} \dot{V}(x) + \alpha V(x) &= 2x^T P(Ax + g(x)) + \alpha x^T P x \\ &= x^T (A^T P + PA + \alpha P)x + 2x^T P z \\ &= \begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned}$$

where $z = g(x)$

z satisfies $z^T z \leq \gamma^2 x^T x$

so we need $P > 0$ and

$$-\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0$$

whenever

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0$$

by S-procedure, this happens if and only if

$$-\begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix} \geq \tau \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$$

for some $\tau \geq 0$

(constraint qualification holds here)

thus, necessary and sufficient conditions for the existence of quadratic Lyapunov function can be expressed as LMI

$$P > 0, \qquad \begin{bmatrix} A^TP + PA + \alpha P + \tau\gamma^2 I & P \\ P & -\tau I \end{bmatrix} \leq 0$$

in variables P, τ (note condition $\tau \geq 0$ is automatic from $2, 2$ block)

by homogeneity, we can write this as

$$P \geq I, \qquad \begin{bmatrix} A^TP + PA + \alpha P + \tau\gamma^2 I & P \\ P & -\tau I \end{bmatrix} \leq 0$$

- solving this LMI to find P is a powerful method
- it beats, for example, solving the Lyapunov equation $A^TP + PA + I = 0$ and hoping the resulting P works

S-procedure for multiple quadratic forms

let $F_0 = F_0^T, \ldots, F_k = F_k^T \in \mathbf{R}^{n \times n}$

when is it true that

$$\text{for all } z, \quad z^T F_1 z \geq 0, \ldots, z^T F_k z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0 \qquad (1)$$

in other words, when does nonnegativity of a set of quadratic forms imply nonnegativity of another?

simple sufficient condition: suppose there are $\tau_1, \ldots, \tau_k \geq 0$, with

$$F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k$$

then for sure the property (1) above holds

(in this case this is only a sufficient condition; it is not necessary) using the matrix inequality condition

$$F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k, \qquad \tau_1, \ldots, \tau_k \geq 0$$

as a sufficient condition for

$$\text{for all } z, \quad z^T F_1 z \geq 0, \ldots, z^T F_k z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$$

is called the (lossy) S-procedure

the matrix inequality condition is an LMI in $\tau_1, \ldots, \tau_k$, therefore easily solved

the constants τ_i are called *multipliers*