Lecture 11
Invariant sets, conservation, and dissipation

- invariant sets
- conserved quantities
- dissipated quantities
- derivative along trajectory
- discrete-time case

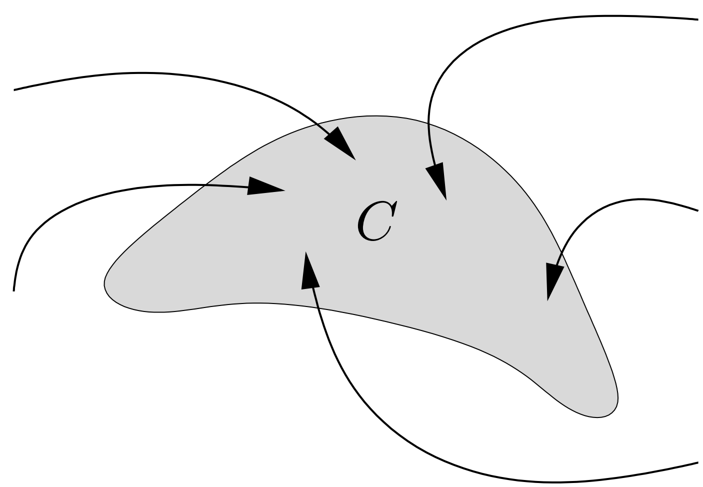
Invariant sets

we consider autonomous, time-invariant nonlinear system $\dot{x} = f(x)$

a set $C \subseteq \mathbf{R}^n$ is *invariant* (w.r.t. system, or f) if for every trajectory x,

$$x(t) \in C \implies x(\tau) \in C \text{ for all } \tau \geq t$$

- if trajectory enters C, or starts in C, it stays in C
- trajectories can cross *into* boundary of C, but never *out* of C



Examples of invariant sets

general examples:

- $\{x_0\}$, where $f(x_0) = 0$ ($i.e.$, x_0 is an equilibrium point)
- any trajectory or union of trajectories, $e.g.$,
  $\{x(t) \mid x(0) \in D,\ t \geq 0,\ \dot{x} = f(x)\}$

more specific examples:

- $\dot{x} = Ax$, $C = \text{span}\{v_1, \ldots, v_k\}$, where $Av_i = \lambda_i v_i$
- $\dot{x} = Ax$, $C = \{z \mid 0 \leq w^T z \leq a\}$, where $w^T A = \lambda w^T$, $\lambda \leq 0$

Invariance of nonnegative orthant

when is nonnegative orthant $\mathbf{R}_+^n$ invariant for $\dot{x} = Ax$?
($i.e.$, when do nonnegative trajectories always stay nonnegative?)

answer: if and only if $A_{ij} \geq 0$ for $i \neq j$

first assume $A_{ij} \geq 0$ for $i \neq j$, and $x(0) \in \mathbf{R}_+^n$; we'll show that $x(t) \in \mathbf{R}_+^n$ for $t \geq 0$

$$x(t) = e^{tA}x(0) = \lim_{k \to \infty} \left(I + (t/k)A\right)^k x(0)$$

for k large enough the matrix $I + (t/k)A$ has all nonnegative entries, so $\left(I + (t/k)A\right)^k x(0)$ has all nonnegative entries

hence the limit above, which is $x(t)$, has nonnegative entries now let's assume that $A_{ij} < 0$ for some $i \neq j$; we'll find trajectory with $x(0) \in \mathbf{R}^n_+$ but $x(t) \notin \mathbf{R}^n_+$ for some $t > 0$

let's take $x(0) = e_j$, so for small $h > 0$, we have $x(h) \approx e_j + hAe_j$

in particular, $x(h)_i \approx hA_{ij} < 0$ for small positive h, *i.e.*, $x(h) \notin \mathbf{R}^n_+$

this shows that if $A_{ij} < 0$ for some $i \neq j$, $\mathbf{R}^n_+$ isn't invariant

Conserved quantities

scalar valued function $\phi : \mathbf{R}^n \to \mathbf{R}$ is called *integral of the motion*, a *conserved quantity*, or *invariant* for $\dot{x} = f(x)$ if for every trajectory x, $\phi(x(t))$ is constant

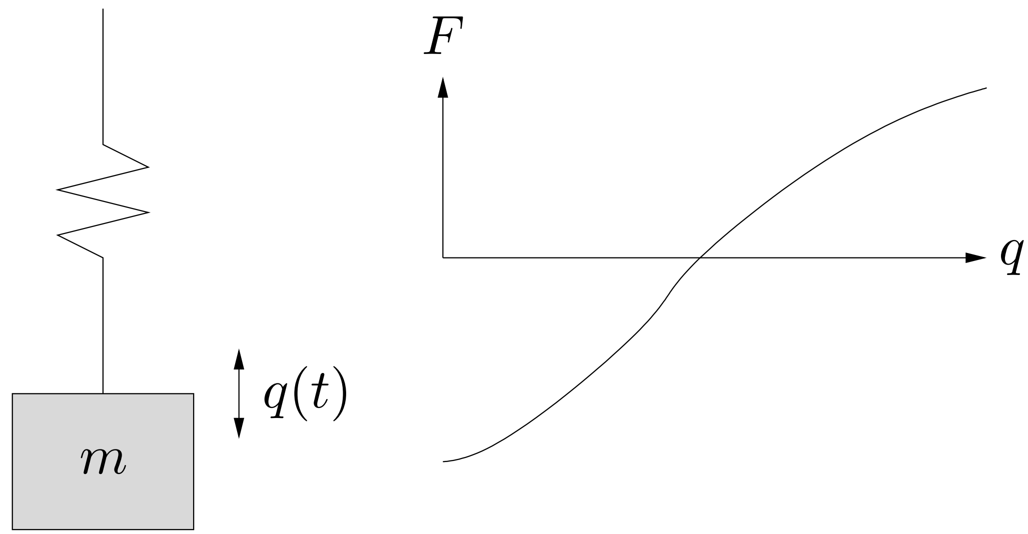
classical examples:

- total energy of a lossless mechanical system
- total angular momentum about an axis of an isolated system
- total fluid in a closed system

level set or *level surface* of ϕ, $\{z \in \mathbf{R}^n \mid \phi(z) = a\}$, are invariant sets

$e.g.$, trajectories of lossless mechanical system stay in surfaces of constant energy

Example: nonlinear lossless mechanical system



$m\ddot{q} = -F = -\phi(q)$, where $m > 0$ is mass, $q(t)$ is displacement, F is restoring force, ϕ is nonlinear spring characteristic with $\phi(0) = 0$

with $x = (q, \dot{q})$, we have

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} x_2 \\ -(1/m)\phi(x_1) \end{bmatrix}$$

potential energy stored in spring is

$$\psi(q) = \int_0^q \phi(u)\, du$$

total energy is kinetic plus potential: $E(x) = (m/2)\dot{q}^2 + \psi(q)$

E is a conserved quantity: if x is a trajectory, then

$$\begin{aligned}
\frac{d}{dt}E(x(t)) &= (m/2)\frac{d}{dt}\dot{q}^2 + \frac{d}{dt}\psi(q) \\
&= m\dot{q}\ddot{q} + \phi(q)\dot{q} \\
&= m\dot{q}(-(1/m)\phi(q)) + \phi(q)\dot{q} \\
&= 0
\end{aligned}$$

$i.e.$, $E(x(t))$ is constant

Derivative of function along trajectory

we have function $\phi : \mathbf{R}^n \to \mathbf{R}$ and $\dot{x} = f(x)$

if x is trajectory of system, then

$$\frac{d}{dt}\phi(x(t)) = D\phi(x(t))\frac{dx}{dt} = \nabla\phi(x(t))^T f(x)$$

we define $\dot{\phi} : \mathbf{R}^n \to \mathbf{R}$ as

$$\dot{\phi}(z) = \nabla\phi(z)^T f(z)$$

intepretation: $\dot{\phi}(z)$ gives $\frac{d}{dt}\phi(x(t))$, if $x(t) = z$

$e.g.$, if $\dot{\phi}(z) > 0$, then $\phi(x(t))$ is increasing when $x(t)$ passes through z

if ϕ is conserved, then $\phi(x(t))$ is constant along any trajectory, so

$$\dot{\phi}(z) = \nabla\phi(z)^T f(x) = 0$$

for all z

this means the vector field $f(z)$ is everywhere orthogonal to $\nabla\phi$, which is normal to the level surface

Dissipated quantities

we say that $\phi : \mathbf{R}^n \to \mathbf{R}$ is a *dissipated quantity* for system $\dot{x} = f(x)$ if for all trajectories, $\phi(x(t))$ is (weakly) decreasing, $i.e.$, $\phi(x(\tau)) \leq \phi(x(t))$ for all $\tau \geq t$

classical examples:

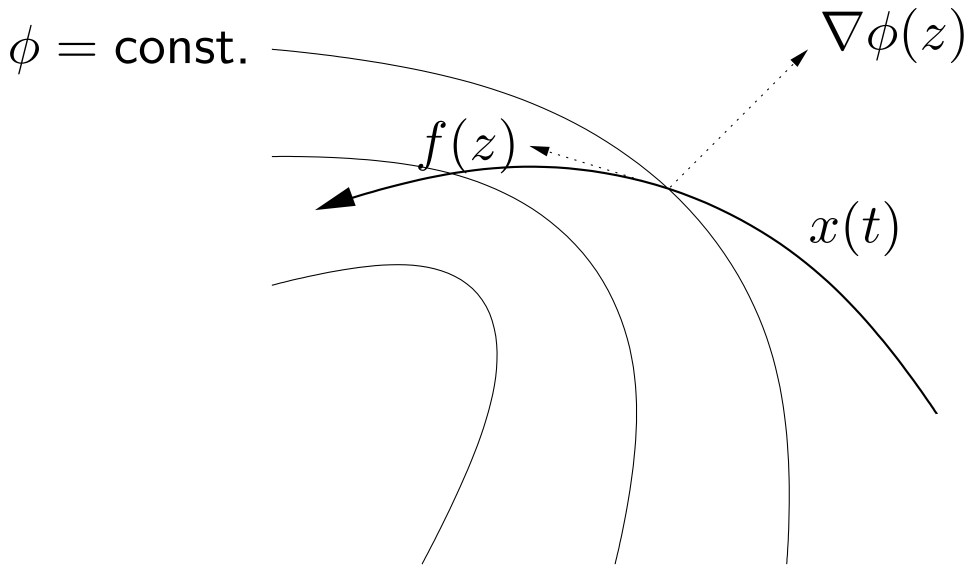
- total energy of a mechanical system with damping
- total fluid in a system that leaks

condition: $\dot{\phi}(z) \leq 0$ for all z, $i.e.$, $\nabla\phi(z)^T f(z) \leq 0$

$-\dot{\phi}$ is sometimes called the *dissipation function*

if ϕ is dissipated quantity, *sublevel sets* $\{z \mid \phi(z) \leq a\}$ are invariant

Geometric interpretation



- vector field points *into* sublevel sets
- $\nabla\phi(z)^T f(z) \leq 0$, *i.e.*, $\nabla\phi$ and f always make an obtuse angle
- trajectories can only "slip down" to lower values of ϕ

Example

linear mechanical system with damping: $M\ddot{q} + D\dot{q} + Kq = 0$

- $q(t) \in \mathbf{R}^n$ is displacement or configuration
- $M = M^T > 0$ is mass or inertia matrix
- $K = K^T > 0$ is stiffness matrix
- $D = D^T \geq 0$ is damping or loss matrix

we'll use state $x = (q, \dot{q})$, so

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$$

consider total (potential plus kinetic) energy

$$E = \frac{1}{2}q^T K q + \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}x^T \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} x$$

we have

$$\begin{aligned} \dot{E}(z) &= \nabla E(z)^T f(z) \\ &= z^T \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} z \\ &= z^T \begin{bmatrix} 0 & K \\ -K & -D \end{bmatrix} z \\ &= -\dot{q}^T D \dot{q} \leq 0 \end{aligned}$$

makes sense: $\frac{d}{dt}$ (total stored energy) $= -$ (power dissipated)

Trajectory limit with dissipated quantity

suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ is dissipated quantity for $\dot{x} = f(x)$

- $\phi(x(t)) \to \phi^*$ as $t \to \infty$, where $\phi^* \in \mathbf{R} \cup \{-\infty\}$
- if trajectory x is bounded and $\dot{\phi}$ is continuous, $x(t)$ converges to the *zero-dissipation set*:

$$x(t) \to \mathcal{D}_0 = \{z \mid \dot{\phi}(z) = 0\}$$

$i.e.$, $\mathbf{dist}\,(x(t), \mathcal{D}_0) \to 0$, as $t \to \infty$ (more on this later)

Linear functions and linear dynamical systems

we consider linear system $\dot{x} = Ax$

when is a linear function $\phi(z) = c^T z$ conserved or dissipated?

$$\dot{\phi} = \nabla\phi(z)^T f(z) = c^T A z$$

$$\dot{\phi}(z) \leq 0 \text{ for all } z \iff \dot{\phi}(z) = 0 \text{ for all } z \iff A^T c = 0$$

$i.e.$, ϕ is dissipated if only if it is conserved, if and only if if $A^T c = 0$
(c is left eigenvector of A with eigenvalue 0)

Quadratic functions and linear dynamical systems

we consider linear system $\dot{x} = Ax$

when is a quadratic form $\phi(z) = z^T P z$ conserved or dissipated?

$$\dot{\phi}(z) = \nabla\phi(z)^T f(z) = 2z^T PAz = z^T(A^T P + PA)z$$

$i.e.$, $\dot{\phi}$ is also a quadratic form

- ϕ is conserved if and only if $A^T P + PA = 0$
  (which means A and $-A$ share at least $\mathbf{Rank}(P)$ eigenvalues)
- ϕ is dissipated if and only if $A^T P + PA \leq 0$

A criterion for invariance

suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ satisfies $\phi(z) = 0 \implies \dot{\phi}(z) < 0$

then the set $C = \{z \mid \phi(z) \leq 0\}$ is invariant

idea: all trajectories on boundary of C cut *into* C, so none can leave

to show this, suppose trajectory x satisfies $x(t) \in C$, $x(s) \notin C$, $t \leq s$

consider (differentiable) function $g : \mathbf{R} \to \mathbf{R}$ given by $g(\tau) = \phi(x(\tau))$

g satisfies $g(t) \leq 0$, $g(s) > 0$

any such function must have at least one point $T \in [t, s]$ where $g(T) = 0$, $g'(T) \geq 0$ (for example, we can take $T = \min\{\tau \geq t \mid g(\tau) = 0\}$)

this means $\phi(x(T)) = 0$ and $\dot{\phi}(x(T)) \geq 0$, a contradiction

Discrete-time systems

we consider nonlinear time-invariant discrete-time system or recursion $x_{t+1} = f(x_t)$

we say $C \subseteq \mathbf{R}^n$ is invariant (with respect to the system) if for every trajectory x,

$$x_t \in C \implies x_\tau \in C \text{ for all } \tau \geq t$$

$i.e.$, trajectories can enter, but cannot leave set C

equivalent to: $z \in C \implies f(z) \in C$

example: when is nonnegative orthant $\mathbf{R}^n_+$ invariant for $x_{t+1} = Ax_t$?

answer: $\Leftrightarrow A_{ij} \geq 0$ for $i,\ j = 1, \ldots, n$

Conserved and dissipated quantities

$\phi : \mathbf{R}^n \to \mathbf{R}$ is conserved under $x_{t+1} = f(x_t)$ if $\phi(x_t)$ is constant, *i.e.*, $\phi(f(z)) = \phi(z)$ for all z

ϕ is a dissipated quantity if $\phi(x_t)$ is (weakly) decreasing, *i.e.*, $\phi(f(z)) \leq \phi(z)$ for all z

we define $\Delta\phi : \mathbf{R}^n \to \mathbf{R}$ by $\Delta\phi(z) = \phi(f(z)) - \phi(z)$

$\Delta\phi(z)$ gives change in ϕ, over one step, starting at z

ϕ is conserved if and only if $\Delta\phi(z) = 0$ for all z

ϕ is dissipated if and only if $\Delta\phi(z) \leq 0$ for all z

Quadratic functions and linear dynamical systems

we consider linear system $x_{t+1} = Ax_t$

when is a quadratic form $\phi(z) = z^T P z$ conserved or dissipated?

$$\Delta\phi(z) = (Az)^T P(Az) - z^T P z = z^T (A^T P A - P) z$$

i.e., $\Delta\phi$ is also a quadratic form

- ϕ is conserved if and only if $A^T P A - P = 0$
  (which means A and A^{-1} share at least $\mathbf{Rank}(P)$ eigenvalues, if A invertible)
- ϕ is dissipated if and only if $A^T P A - P \leq 0$