Lecture 4
Continuous time linear quadratic regulator

- continuous-time LQR problem
- dynamic programming solution
- Hamiltonian system and two point boundary value problem
- infinite horizon LQR
- direct solution of ARE via Hamiltonian

Continuous-time LQR problem

continuous-time system $\dot{x} = Ax + Bu$, $x(0) = x_0$

problem: choose $u : [0, T] \to \mathbf{R}^m$ to minimize

$$J = \int_0^T \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) \, d\tau + x(T)^T Q_f x(T)$$

- T is *time horizon*
- $Q = Q^T \geq 0$, $Q_f = Q_f^T \geq 0$, $R = R^T > 0$ are *state cost*, *final state cost*, and *input cost* matrices

. . . an *infinite-dimensional problem*: (trajectory $u : [0, T] \to \mathbf{R}^m$ is variable)

Dynamic programming solution

we'll solve LQR problem using dynamic programming

for $0 \leq t \leq T$ we define the **value function** $V_t : \mathbf{R}^n \to \mathbf{R}$ by

$$V_t(z) = \min_u \int_t^T \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) \, d\tau + x(T)^T Q_f x(T)$$

subject to $x(t) = z$, $\dot{x} = Ax + Bu$

- minimum is taken over all possible signals $u : [t, T] \to \mathbf{R}^m$
- $V_t(z)$ gives the minimum LQR cost-to-go, starting from state z at time t
- $V_T(z) = z^T Q_f z$

fact: V_t is quadratic, *i.e.*, $V_t(z) = z^T P_t z$, where $P_t = P_t^T \geq 0$

similar to discrete-time case:

- P_t can be found from a *differential equation* running backward in time from $t = T$
- the LQR optimal u is easily expressed in terms of P_t

we start with $x(t) = z$

let's take $u(t) = w \in \mathbf{R}^m$, a constant, over the time interval $[t, t+h]$, where $h > 0$ is small

cost incurred over $[t, t+h]$ is

$$\int_t^{t+h} \left(x(\tau)^T Q x(\tau) + w^T R w \right) \, d\tau \approx h(z^T Q z + w^T R w)$$

and we end up at $x(t+h) \approx z + h(Az + Bw)$

min-cost-to-go from where we land is approximately

$$
\begin{array}{rl}
& V_{t+h}(z + h(Az + Bw)) \\
= & (z + h(Az + Bw))^T P_{t+h}(z + h(Az + Bw)) \\
\approx & (z + h(Az + Bw))^T (P_t + h\dot{P}_t)(z + h(Az + Bw)) \\
\approx & z^T P_t z + h\left((Az + Bw)^T P_t z + z^T P_t (Az + Bw) + z^T \dot{P}_t z\right)
\end{array}
$$

(dropping h^2 and higher terms)

cost incurred plus min-cost-to-go is approximately

$$
z^T P_t z + h\left(z^T Q z + w^T R w + (Az + Bw)^T P_t z + z^T P_t (Az + Bw) + z^T \dot{P}_t z\right)
$$

minimize over w to get (approximately) optimal w:

$$
2hw^T R + 2hz^T P_t B = 0
$$

so

$$w^* = -R^{-1}B^TP_tz$$

thus optimal u is time-varying linear state feedback:

$$u_{\rm lqr}(t) = K_tx(t), \qquad K_t = -R^{-1}B^TP_t$$

HJ equation

now let's substitute w^* into HJ equation:

$$z^T P_t z \approx z^T P_t z +$$
$$+h \left(z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P_t z + z^T P_t (Az + Bw^*) + z^T \dot{P}_t z \right)$$

yields, after simplification,

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q$$

which is the *Riccati differential equation* for the LQR problem

we can solve it (numerically) using the *final condition* $P_T = Q_f$

Summary of cts-time LQR solution via DP

1. solve Riccati differential equation

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q, \qquad P_T = Q_f$$

(backward in time)

2. optimal u is $u_{\mathrm{lqr}}(t) = K_t x(t)$, $K_t := -R^{-1} B^T P_t$

DP method readily extends to time-varying A, B, Q, R, and tracking problem

Steady-state regulator

usually P_t rapidly converges as t decreases below T

limit $P_{\rm ss}$ satisfies (cts-time) algebraic Riccati equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

a quadratic matrix equation

- $P_{\rm ss}$ can be found by (numerically) integrating the Riccati differential equation, or by direct methods
- for t not close to horizon T, LQR optimal input is approximately a linear, constant state feedback

$$u(t) = K_{\rm ss}x(t), \qquad K_{\rm ss} = -R^{-1}B^T P_{\rm ss}$$

Derivation via discretization

let's discretize using small step size $h > 0$, with $Nh = T$

$$x((k+1)h) \approx x(kh) + h\dot{x}(kh) = (I + hA)x(kh) + hBu(kh)$$

$$J \approx \frac{h}{2} \sum_{k=0}^{N-1} \left(x(kh)^T Q x(kh) + u(kh)^T R u(kh) \right) + \frac{1}{2} x(Nh)^T Q_f x(Nh)$$

this yields a discrete-time LQR problem, with parameters

$$\tilde{A} = I + hA, \qquad \tilde{B} = hB, \qquad \tilde{Q} = hQ, \qquad \tilde{R} = hR, \qquad \tilde{Q}_f = Q_f$$

solution to discrete-time LQR problem is $u(kh) = \tilde{K}_k x(kh)$,

$$\tilde{K}_k = -(\tilde{R} + \tilde{B}^T \tilde{P}_{k+1} \tilde{B})^{-1} \tilde{B}^T \tilde{P}_{k+1} \tilde{A}$$

$$\tilde{P}_{k-1} = \tilde{Q} + \tilde{A}^T \tilde{P}_k \tilde{A} - \tilde{A}^T \tilde{P}_k \tilde{B} (\tilde{R} + \tilde{B}^T \tilde{P}_k \tilde{B})^{-1} \tilde{B}^T \tilde{P}_k \tilde{A}$$

substituting and keeping only h^0 and h^1 terms yields

$$\tilde{P}_{k-1} = hQ + \tilde{P}_k + hA^T \tilde{P}_k + h\tilde{P}_k A - h\tilde{P}_k B R^{-1} B^T \tilde{P}_k$$

which is the same as

$$-\frac{1}{h}(\tilde{P}_k - \tilde{P}_{k-1}) = Q + A^T \tilde{P}_k + \tilde{P}_k A - \tilde{P}_k B R^{-1} B^T \tilde{P}_k$$

letting $h \to 0$ we see that $\tilde{P}_k \to P_{kh}$, where

$$-\dot{P} = Q + A^T P + PA - PBR^{-1}B^T P$$

similarly, we have

$$\begin{aligned}
\tilde{K}_k &= -(\tilde{R} + \tilde{B}^T \tilde{P}_{k+1} \tilde{B})^{-1} \tilde{B}^T \tilde{P}_{k+1} \tilde{A} \\
&= -(hR + h^2 B^T \tilde{P}_{k+1} B)^{-1} h B^T \tilde{P}_{k+1} (I + hA) \\
&\to -R^{-1} B^T P_{kh}
\end{aligned}$$

as $h \to 0$

Derivation using Lagrange multipliers

pose as constrained problem:

$$\begin{array}{ll} \text{minimize} & J = \frac{1}{2}\int_0^T x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau + \frac{1}{2} x(T)^T Q_f x(T) \\ \text{subject to} & \dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, T] \end{array}$$

- optimization variable is *function* $u : [0, T] \to \mathbf{R}^m$
- infinite number of equality constraints, one for each $t \in [0, T]$

introduce Lagrange multiplier *function* $\lambda : [0, T] \to \mathbf{R}^n$ and form

$$L = J + \int_0^T \lambda(\tau)^T (Ax(\tau) + Bu(\tau) - \dot{x}(\tau))\, d\tau$$

Optimality conditions

(note: you need *distribution theory* to really make sense of the derivatives here . . .)

from $\nabla_{u(t)} L = Ru(t) + B^T\lambda(t) = 0$ we get $u(t) = -R^{-1}B^T\lambda(t)$

to find $\nabla_{x(t)} L$, we use

$$\int_0^T \lambda(\tau)^T \dot{x}(\tau)\, d\tau = \lambda(T)^T x(T) - \lambda(0)^T x(0) - \int_0^T \dot{\lambda}(\tau)^T x(\tau)\, d\tau$$

from $\nabla_{x(t)} L = Qx(t) + A^T\lambda(t) + \dot{\lambda}(t) = 0$ we get

$$\dot{\lambda}(t) = -A^T\lambda(t) - Qx(t)$$

from $\nabla_{x(T)} L = Q_f x(T) - \lambda(T) = 0$, we get $\lambda(T) = Q_f x(T)$

Co-state equations

optimality conditions are

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \qquad \dot{\lambda} = -A^T\lambda - Qx, \quad \lambda(T) = Q_f x(T)$$

using $u(t) = -R^{-1}B^T\lambda(t)$, can write as

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

- $2n \times 2n$ matrix above is called *Hamiltonian* for problem
- with conditions $x(0) = x_0$, $\lambda(T) = Q_f x(T)$, called *two-point boundary value problem*

as in discrete-time case, we can show that $\lambda(t) = P_t x(t)$, where

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q, \qquad P_T = Q_f$$

in other words, value function P_t gives simple relation between x and λ

to show this, we show that $\lambda = Px$ satisfies co-state equation
$\dot{\lambda} = -A^T \lambda - Qx$

$$\begin{aligned}
\dot{\lambda} &= \frac{d}{dt}(Px) = \dot{P}x + P\dot{x} \\
&= -(Q + A^T P + PA - PBR^{-1}B^T P)x + P(Ax - BR^{-1}B^T \lambda) \\
&= -Qx - A^T Px + PBR^{-1}B^T Px - PBR^{-1}B^T Px \\
&= -Qx - A^T \lambda
\end{aligned}$$

Solving Riccati differential equation via Hamiltonian

the (quadratic) Riccati differential equation

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q$$

and the (linear) Hamiltonian differential equation

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

are closely related

$\lambda(t) = P_t x(t)$ suggests that P should have the form $P_t = \lambda(t)x(t)^{-1}$ (but this doesn't make sense unless x and λ are scalars) consider the Hamiltonian *matrix* (linear) differential equation

$$\frac{d}{dt}\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

where $X(t),\ Y(t) \in \mathbf{R}^{n\times n}$

then, $Z(t) = Y(t)X(t)^{-1}$ satisfies Riccati differential equation

$$-\dot{Z} = A^TZ + ZA - ZBR^{-1}B^TZ + Q$$

hence we can solve Riccati DE by solving (linear) matrix Hamiltonian DE, with final conditions $X(T) = I$, $Y(T) = Q_f$, and forming $P(t) = Y(t)X(t)^{-1}$

$$
\begin{aligned}
\dot{Z} &= \frac{d}{dt} Y X^{-1} \\
&= \dot{Y} X^{-1} - Y X^{-1} \dot{X} X^{-1} \\
&= (-QX - A^T Y) X^{-1} - Y X^{-1} \left(AX - BR^{-1} B^T Y \right) X^{-1} \\
&= -Q - A^T Z - ZA + ZBR^{-1} B^T Z
\end{aligned}
$$

where we use two identities:

- $\dfrac{d}{dt}(F(t)G(t)) = \dot{F}(t)G(t) + F(t)\dot{G}(t)$
- $\dfrac{d}{dt}\left(F(t)^{-1}\right) = -F(t)^{-1}\dot{F}(t)F(t)^{-1}$

Infinite horizon LQR

we now consider the infinite horizon cost function

$$J = \int_0^\infty x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\; d\tau$$

we define the value function as

$$V(z) = \min_u \int_0^\infty x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\; d\tau$$

subject to $x(0) = z$, $\dot{x} = Ax + Bu$

we assume that (A, B) is controllable, so V is finite for all z

can show that V is quadratic: $V(z) = z^T P z$, where $P = P^T \geq 0$

optimal u is $u(t) = Kx(t)$, where $K = -R^{-1}B^TP$
($i.e.$, a constant linear state feedback)

HJ equation is ARE

$$Q + A^TP + PA - PBR^{-1}B^TP = 0$$

which together with $P \geq 0$ characterizes P

can solve as limiting value of Riccati DE, or via direct method

Closed-loop system

with K LQR optimal state feedback gain, closed-loop system is

$$\dot{x} = Ax + Bu = (A + BK)x$$

fact: closed-loop system is stable when (Q, A) observable and (A, B) controllable

we denote eigenvalues of $A + BK$, called *closed-loop eigenvalues*, as $\lambda_1, \ldots, \lambda_n$

with assumptions above, $\Re\lambda_i < 0$

Solving ARE via Hamiltonian

$$\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^T P \\ -Q - A^T P \end{bmatrix} = \begin{bmatrix} A + BK \\ -Q - A^T P \end{bmatrix}$$

and so

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A + BK & -BR^{-1}B^T \\ 0 & -(A + BK)^T \end{bmatrix}$$

where 0 in lower left corner comes from ARE

note that

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$$

we see that:

- eigenvalues of Hamiltonian H are $\lambda_1, \ldots, \lambda_n$ and $-\lambda_1, \ldots, -\lambda_n$
- hence, closed-loop eigenvalues are the eigenvalues of H with negative real part

let's assume $A + BK$ is diagonalizable, *i.e.*,

$$T^{-1}(A + BK)T = \Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

then we have $T^T(-A - BK)^T T^{-T} = -\Lambda$, so

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & T^T \end{bmatrix} \begin{bmatrix} A + BK & -BR^{-1}B^T \\ 0 & -(A + BK)^T \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-T} \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda & -T^{-1}BR^{-1}B^T T^{-T} \\ 0 & -\Lambda \end{bmatrix}$$

putting it together we get

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & T^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-T} \end{bmatrix}$$

$$= \begin{bmatrix} T^{-1} & 0 \\ -T^TP & T^T \end{bmatrix} H \begin{bmatrix} T & 0 \\ PT & T^{-T} \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda & -T^{-1}BR^{-1}B^TT^{-T} \\ 0 & -\Lambda \end{bmatrix}$$

and so

$$H \begin{bmatrix} T \\ PT \end{bmatrix} = \begin{bmatrix} T \\ PT \end{bmatrix} \Lambda$$

thus, the n columns of $\begin{bmatrix} T \\ PT \end{bmatrix}$ are the eigenvectors of H associated with the stable eigenvalues $\lambda_1, \ldots, \lambda_n$

Solving ARE via Hamiltonian

- find eigenvalues of H, and let $\lambda_1, \ldots, \lambda_n$ denote the n stable ones (there are exactly n stable and n unstable ones)

- find associated eigenvectors $v_1, \ldots, v_n$, and partition as

$$\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right] = \left[\begin{array}{c} X \\ Y \end{array}\right] \in \mathbf{R}^{2n \times n}$$

- $P = YX^{-1}$ is unique PSD solution of the ARE

(this is very close to the method used in practice, which does not require $A + BK$ to be diagonalizable)