EE276: Problem Session #4 Problems

1. **Encoder and decoder as part of the channel:** Consider a binary symmetric channel with crossover probability 0.1. A possible coding scheme for this channel with two codewords of length 3 is to encode message a_1 as 000 and a_2 as 111. With this coding scheme, we can consider the combination of encoder, channel and decoder as forming a new BSC, with two inputs a_1 and a_2 and two outputs a_1 and a_2.

   (a) Calculate the crossover probability of this channel.

   (b) What is the capacity of this channel in bits per transmission of the original channel?

   (c) What is the capacity of the original BSC with crossover probability 0.1?

   (d) Prove a general result that for any channel, considering the encoder, channel and decoder together as a new channel from messages to estimated messages will not increase the capacity in bits per transmission of the original channel.

2. **Channel with uniformly distributed noise:** Consider a additive channel whose input alphabet $= \{0, \pm 1, \pm 2\}$, and whose output $Y = X + Z$, where Z is uniformly distributed over the interval $[-1, 1]$. Thus the input of the channel is a discrete random variable, while the output is continuous. Calculate the capacity $C = \max_{p(x)} I(X;Y)$ of this channel.

3. **Entropy rates of Markov chains.**

   (a) Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1-p & p \\ 1 & 0 \end{bmatrix}$$

   starting at its stationary distribution.

   (b) Find the maximum value of the entropy rate of the Markov chain of part (a). We expect that the maximizing value of p should be less than $1/2$, since the 0 state permits more information to be generated than the 1 state.

   (c) Let $N(t)$ be the number of allowable state sequences of length t for the Markov chain of part (a). Find $N(t)$ and calculate

$$H_0 = \lim_{t \to \infty} \frac{1}{t} \log N(t) \,.$$

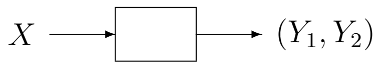
   Hint: Find a linear recurrence that expresses $N(t)$ in terms of $N(t-1)$ and $N(t-2)$. Why is H_0 an upper bound on the entropy rate of the Markov chain? Compare H_0 with the maximum entropy found in part (b).

4. **A channel with two independent looks at Y.**

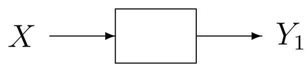
   Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X.

(a) Show $I(X;Y_1,Y_2) = 2I(X;Y_1) - I(Y_1,Y_2)$.

(b) Show that the capacity C_2 of the channel



is less than twice the capacity C_1 of the channel



(c) Show that C_2 is at least C_1.

(d) When is C_2 equal to C_1?