EE276: Problem Session #2

1. **Conditional mutual information vs. unconditional mutual information.** Give examples of joint random variables X, Y and Z such that

   (a) $I(X;Y \mid Z) < I(X;Y)$,

   (b) $I(X;Y \mid Z) > I(X;Y)$.

2. **Calculation of typical set.** To clarify the notion of a typical set $A_\epsilon^{(n)}$ and the smallest set of high probability $B_\delta^{(n)}$, we will calculate the set for a simple example. Consider a sequence of i.i.d. binary random variables, $X_1, X_2, \ldots, X_n$, where the probability that $X_i = 1$ is 0.6 (and therefore the probability that $X_i = 0$ is 0.4).

k	$\binom{n}{k}$	$\sum_{j \le k} \binom{n}{j}$	$p(x^n) = p^k(1-p)^{n-k}$	$\binom{n}{k} p^k (1-p)^{n-k}$	Cumul. pr.	$-\frac{1}{n} \log p(x^n)$
0	1	1	1.125898×10^{-10}	0.000000	0.000000	1.321928
1	25	26	1.688847×10^{-10}	0.000000	0.000000	1.298530
2	300	326	2.533271×10^{-10}	0.000000	0.000000	1.275131
3	2300	2626	3.799908×10^{-10}	0.000000	0.000001	1.251733
4	12650	15276	5.699862×10^{-10}	0.000007	0.000008	1.228334
5	53130	68406	8.549795×10^{-10}	0.000045	0.000054	1.204936
6	177100	245506	1.282469×10^{-09}	0.000227	0.000281	1.181537
7	480700	726206	1.923704×10^{-09}	0.000925	0.001205	1.158139
8	1081575	1807781	2.885556×10^{-09}	0.003121	0.004326	1.134740
9	2042975	3850756	4.328335×10^{-09}	0.008843	0.013169	1.111342
10	3268760	7119516	6.492503×10^{-09}	0.021222	0.034392	1.087943
11	4457400	11576916	9.738756×10^{-09}	0.043410	0.077801	1.064545
12	5200300	16777216	1.460813×10^{-08}	0.075967	0.153768	1.041146
13	5200300	21977516	2.191220×10^{-08}	0.113950	0.267718	1.017748
14	4457400	26434916	3.286831×10^{-08}	0.146507	0.414225	0.994349
15	3268760	29703676	4.930247×10^{-08}	0.161158	0.575383	0.970951
16	2042975	31746651	7.395371×10^{-08}	0.151086	0.726468	0.947552
17	1081575	32828226	1.109306×10^{-07}	0.119980	0.846448	0.924154
18	480700	33308926	1.663959×10^{-07}	0.079986	0.926435	0.900755
19	177100	33486026	2.495939×10^{-07}	0.044203	0.970638	0.877357
20	53130	33539156	3.743908×10^{-07}	0.019891	0.990529	0.853958
21	12650	33551806	5.615863×10^{-07}	0.007104	0.997633	0.830560
22	2300	33554106	8.423795×10^{-07}	0.001937	0.999571	0.807161
23	300	33554406	1.263569×10^{-06}	0.000379	0.999950	0.783763
24	25	33554431	1.895354×10^{-06}	0.000047	0.999997	0.760364
25	1	33554432	2.843032×10^{-06}	0.000003	1.000000	0.736966

   (a) Calculate $H(X)$.

   (b) With $n = 25$ and $\epsilon = 0.1$, which sequences fall in the typical set $A_\epsilon^{(n)}$? What is the probability of the typical set? How many elements are there in the typical set? (This involves computation of a table of probabilities for sequences with k 1's, $0 \leq k \leq 25$, and finding those sequences that are in the typical set.)

(c) What is the ratio of the probabilities of the two elements with the highest and lowest probabilities in the typical set $A_\epsilon^{(n)}$ for $n = 25$ and $\epsilon = 0.1$? What happens to this ratio as n grows? Give an upper bound on this ratio.

Note that even though the probabilities of the sequences in the typical set can be very different, number of bits to represent $A_\epsilon^{(n)}$ per symbol gives a very accurate estimation of the optimal compression rate.

3. **Random box size.** An n-dimensional rectangular box with sides $X_1, X_2, X_3, \ldots, X_n$ is to be constructed. The volume is $V_n = \prod_{i=1}^n X_i$. The edge length l of a n-cube with the same volume as the random box is $l_n = V_n^{1/n}$. Let $X_1, X_2, \ldots$ be i.i.d. uniform random variables over the unit interval $[0, 1]$. Show that the random variable l_n converges to $1/e$ in probability. How does this compare to the expected edge length $E(X)$?

Note: To show convergence in probability, we want to show that for any $\epsilon > 0$,

$$P(\{|l_n - \frac{1}{e}| > \epsilon\}) = 0. \tag{1}$$

4. **AEP and mutual information**. Let (X_i, Y_i) be i.i.d. $\sim p(x, y)$. We form the log likelihood ratio of the hypothesis that X and Y are independent vs. the hypothesis that X and Y are dependent. What is the limit of

$$\frac{1}{n} \log \frac{p(X^n)p(Y^n)}{p(X^n, Y^n)}?$$